CATEGORICAL ALGEBRA

by

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Foreword

These notes are based on lectures given at Göteborgs Universitet during spring 1968. They are aimed at giving a fairly elementary and detailed introduction to the basic concepts of category theory and thus prepare the reader for the study Mitchell's monograph [18], Gabriel's paper on abelian categories [10], etc. A second volume of notes with emphasis on the theory of Grothendieck categories will, hopefully, appear later.

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Introduction

Instead of trying to give an account of the subject matter of category theory we will just mention a few examples which illustrate the applications of the theory of categories and functors.

A) Homology theory: The purpose of algebraic topology is to carry over as much as possible of topology to algebra, since algebra lends itself better to computations. An example of such a transition from topology to algebra is given by homology theories. A homology theory H assigns to each topological space X an abelian group $H(X)$ which is called the "homology group" of X, and furthermore assigns to each continuous map $f: X \to Y$ of topological spaces a corresponding group homomorphism $H(f): H(X) \to H(Y)$. This assignment should satisfy the following conditions in Eilenberg - Steenrod's axiomatization of homology theory:

1) $H(id_X) = id_{H(X)}$.
2) $H(gf) = H(g) \cdot H(f)$ when $f: X \to Y$ and $g: Y \to Z$.
3) - 7) see [8], Ch. I, § 3.

We are here only interested in the first two axioms, which guarantee the "naturality" of H. In functor terminology they say that the assignment $X \to H(X)$, $f \to H(f)$ is a functor from the category of topological spaces and continuous maps to the category of abelian groups and group homomorphisms. The categorical terminology is useful when describing homology theories, but it is hardly indispensable. It is when we want to compare homology theories with each other that we feel the need of the functorial language.

Let H and K be two homology theories. When comparing H and K we use the notion of a natural transformation $\Phi: H \to K$. Such a transformation is obtained if for every space X there is defined a group homomorphism $\Phi_X: H(X) \to K(X)$ which is natural in the sense that if $f: X \to Y$ is a continuous map, then the following diagram is commutative:

$$\begin{array}{ccccc} & H(X) & \xrightarrow{\Phi_X} & K(X) & \\ H(f) & \downarrow & & \downarrow & K(f) \\ & H(Y) & \xrightarrow[\Phi_Y]{} & K(Y) & \end{array}$$

Φ is called a natural equivalence if furthermore every Φ_X is an isomorphism.

B) There exist numerous other examples of functors which yield a transition from one "theory" to another, e.g.:

(i) to each affine variety is assigned its coordinate ring.

(ii) to each commutative ring is assigned the topological space consisting of the set of prime ideals of the ring, with its Zariski topology.

(iii) to each compact Hausdorff space is assigned its ring of continuous complex-valued functions, which gives a functor into the category of commutative Banach algebras.

In each of these examples we should have specified what the functor does to the morphisms, but it should be clear from the contexts what the morphisms are and where they go under the functor.

C) Homological algebra (i.e. the study of projective or injective resolutions, derived functors, etc.) is usually pursued in categories of modules (e.g. as in [6]). But in algebraic geometry one needs to do similar things for sheaves of abelian groups. By doing homological algebra in abelian categories one gets a unifying treatment of the subject ([12], [17]).

D) Adjoint functors. For each abelian group A we let $G(A)$ denote the underlying set of A. G may be considered as a "forgetful" functor from the category (Ab) of abelian groups to the category (Ens) of sets. Conversely we may to each set S assign the free abelian group $F(S)$ on S, and to each set map $S \to S'$ assign the homomorphism $F(S) \to F(S')$ obtained by linear extension. In this way we obtain a functor F from (Ens) to (Ab), The two functors are interrelated by the formula

$$\mathrm{Hom}(S, G(A)) \cong \mathrm{Hom}(F(S), A)$$

where the left Hom denotes the set of functions $S \to G(A)$, while the right Hom denotes the set of group homomorphisms $F(S) \to A$. This formula is an example of a very common phenomenon in mathematics and is described by saying that G is a right adjoint of F. The theory of adjoint functors was initiated by Kan [13] and is one of the fundamental tools of category theory.

Chapter I. Categories: Inside theory.

§ 1. Categories.

It is clear from the examples given in the Introduction that a category should consist of "objects" and "morphisms". But the objects are uniquely determined by their identity morphisms and hence they may be left out in the definition of categories. We will therefore first make an abstract, "algebraic", definition of categories and then see how this may be reformulated in terms of objects and morphisms.

Definition. A category $\underline{C}$ is a set $\underline{M}$ with a binary composition defined for certain pairs in $\underline{M}$, satisfying:

C 1: If either $\gamma(\beta\alpha)$ or $(\gamma\beta)\alpha$ is defined, then both are defined and equal.

C 2: If $\beta\alpha$ and $\gamma\beta$ are defined and β is an identity (i.e. $\xi\beta = \xi$ and $\beta\eta = \eta$ whenever defined), then $\gamma\alpha$ is defined.

C 3: For each α there exists identities e_l and e_r such that $e_l\alpha$ and αe_r are defined.

Lemma 1. The identities e_l and e_r in C 3 are uniquely determined by α.

Proof. Suppose also e' is an identity and $e'\alpha$ is defined. Then $e_l(e'\alpha) = e_l\alpha$ is defined, so also e_le' is defined by C 1. Both

e_1 and e' are identities, so $e_1 = e_1 e' = e'$.

Let $\underline{C}$ be a category. Let the set of identities in $\underline{C}$ be indexed by a set $\underline{O}$ and denote the identity corresponding to $A \in \underline{O}$ by 1_A. (E.g. we may choose $\underline{O}$ to be the set of identities itself.)

For each $\alpha \in \underline{M}$ there exist uniquely determined A and B in $\underline{O}$ such that $\alpha 1_A$ and $1_B \alpha$ are defined (C3 and Lemma 1). α is then said to be a <u>morphism</u> from A to B, and we write $\alpha: A \to B$. The set of morphisms from A to B is denoted by $\mathrm{Hom}_{\underline{C}}(A,B)$, and then $\underline{M} = \bigcup_{(A,B)} \mathrm{Hom}(A,B)$ (disjoint union). The elements of $\underline{O}$ are called <u>objects</u> and one often writes $\mathrm{Ob}(\underline{C})$ for $\underline{O}$. Similarly one sometimes writes $\mathrm{Mor}(\underline{C})$ for the underlying set $\underline{M}$ of $\underline{C}$.

<u>Lemma 2</u>. Let $\alpha: A \to B$ and $\beta: C \to D$ be morphisms. Then $\beta\alpha$ is defined if and only if $B = C$.

<u>Proof</u>. If $B = C$, then $\beta\alpha$ is defined by C 2. Conversely, if $\beta\alpha$ is defined, then $\beta\alpha = \beta(1_B \alpha)$ and $\beta 1_B$ is defined by C 1. Hence $B = C$, by Lemma 1.

It is now quite easy to see that the following definition of categories is equivalent to the one given above (which definition one uses depends on what applications one has in mind):

<u>"Definition"</u>. A <u>category</u> consists of:

i) a set $\underline{O}$, whose elements are called objects;

ii) a set $\mathrm{Hom}(A,B)$, whose elements are called <u>morphisms from A to B</u>, for each ordered pair (A,B) of objects;

iii) a composition $\mathrm{Hom}(B,C) \times \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$ for each ordered triple (A,B,C);

and satisfies:

1) $\mathrm{Hom}(A,B)$ and $\mathrm{Hom}(C,D)$ are disjoint sets if $(A,B) \neq (C,D)$;

2) $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ when these compositions of morphisms are defined;

3) for each object A there exists $1_A \in \mathrm{Hom}(A,A)$ such that $1_A\alpha = \alpha$ and $\beta 1_A = \beta$ when these compositions are defined.

Examples of categories:

1. A category with only one identity (i.e. only one object) is called a <u>monoid</u> (or <u>semigroup with identity</u>).

2. Recall that a set $\underline{O}$ is called preordered under a binary relation $\leq$ if the relation is reflexive and transitive. $\underline{O}$ is partially ordered under $\leq$ if furthermore $x \leq y$, $y \leq x \Rightarrow x = y$. Every preordered set $\underline{O}$ defines a category with $\underline{O}$ as the set of objects, and Hom(A,B) consisting of one element if $A \leq B$ and being empty if $A \not\leq B$. Conversely, a category where each Hom (A,B) consists of at most one element defines naturally a preordered set.
3. For each category $\underline{C}$ there exists a dual category $\underline{C}^o$, whose underlying set is the same as that of $\underline{C}$ but with $\alpha * \beta = \beta \cdot \alpha$, where $*$ denotes composition in $\underline{C}^o$ and $\cdot$ composition in $\underline{C}$ (so $\underline{C}^o$ is obtained by "reversing the arrows" of $\underline{C}$). Every definition or theorem for the category $\underline{C}$ may be dualized to a corresponding definition or theorem for $\underline{C}^o$. It is not necessary here to develop any meta-theory for the dualization procedure, since we will meet no difficulties in the future when deciding what the dual theorems and proofs look like. If a result for the category $\underline{C}$ is numbered as Prop. n, then the dual result for $\underline{C}^o$ will be refered to as Prop. n^*. (A more detailed discussion of duality may be found in [4]).

When trying to define the category of all sets one meets the difficulty that there exists no "sets of all sets". Therefore we must restrict ourselves to consider only sets belonging to some sufficiently large set (a so called "universe"; alternatively we could have defined categories to be classes instead of sets).

Definition. A set $\underline{U}$ is called a universe if:

U 1: If $X \in \underline{U}$, then $X \subset \underline{U}$.

U 2: If $X \in \underline{U}$, then $2^X \in \underline{U}$.

U 3: If $X, Y \in \underline{U}$, then $\{X,Y\} \in \underline{U}$.

U 4: If $(X_i)_{i \in I}$ is a family where $I \in \underline{U}$ and each $X_i \in \underline{U}$, then $\bigcup_I X_i \in \underline{U}$.

We introduce the following

Set - theoretical axiom: Every set is a member of some universe.

In [15] it is proved that this axiom is equivalent to requiring that for each cardinal number α there exists an inaccessible cardinal number β such that $\alpha < \beta$.

Let $\underline{U}$ be a universe. A category $\underline{C}$ is called a $\underline{U}$-category if Hom $(A,B) \in \underline{U}$ for each pair of objects A,B. $\underline{C}$ is called $\underline{U}$-small (or just small if the universe is fixed throughout the argument) if $\underline{C} \in \underline{U}$. Note that $\underline{C}$ is $\underline{U}$-small if and only if $\underline{C}$ is a $\underline{U}$-category and $\mathrm{Ob}(\underline{C}) \in \underline{U}$. It should also be noticed that given any category $\underline{C}$, there exists a smallest universe $\underline{U}$ such that $\underline{C}$ is $\underline{U}$-small (this is a consequence of the set-theoretical axiom and the fact that any intersection of universes is a universe).

Examples:

4. $\underline{U}$-(Ens) is the category whose objects are the sets belonging to $\underline{U}$ and where Hom (A,B) consists of all functions from A to B, with the ordinary composition.
5. $\underline{U}$-(Top) has as objects all topological spaces in $\underline{U}$ (i.e. whose underlying sets belong to $\underline{U}$), and as morphisms all continuous functions between such spaces.
6. $\underline{U}$-(Gr), resp. $\underline{U}$-(Ab), has as objects all groups in $\underline{U}$, resp. all abelian groups in $\underline{U}$, and as morphisms all group homomorphisms.
7. U-(Banach) has as objects all Banach spaces in $\underline{U}$ and as morphisms all continuous linear functions between these spaces.
8. The category $\underline{U}$-(Ens)$_0$ of sets with base-points has as objects all pairs (A,a), where A is a set in $\underline{U}$ and $a \in A$. A morphism $(A,a) \to (B,b)$ is a function $A \to B$ which maps a to b. Similarly one defines $\underline{U}$-(Top)$_0$.

All these categories are $\underline{U}$-categories, but they are not $\underline{U}$-small. In the future we will often drop the "$\underline{U}$" and just write (Ens), (Ab), etc, since the universe will be kept fixed.

Definition. Let $\underline{C}$ be a category with underlying set $\underline{M}$. A subset $\underline{N}$ of $\underline{M}$ defines a subcategory $\underline{D}$ of $\underline{C}$ if:

1) $\underline{N}$ is closed under the composition in $\underline{C}$.
2) If $\alpha \in \underline{N}$ and e is an identity in $\underline{C}$ such that either αe or $e\alpha$ is defined, then $e \in \underline{N}$.

$\underline{D}$ is then the category on $\underline{N}$ obtained by restricting the binary composition in $\underline{C}$ to $\underline{N}$.

Using instead the "objective" approach to categories we may formulate the definition of subcategories as follows:

"Definition". A category $\underline{D}$ is a subcategory of a category $\underline{C}$ if:

1) $Ob(\underline{D}) \subset Ob(\underline{C})$.
2) $Hom_{\underline{D}}(A,B) \subset Hom_{\underline{C}}(A,B)$ when A,B are objects in $\underline{D}$.
3) The composition in $\underline{D}$ is induced by the composition in $\underline{C}$.
4) 1_A is the same in $\underline{D}$ as in $\underline{C}$, when A is an object in $\underline{D}$.

The subcategory is full if equality holds in 2).

Example:

9. (Ab) is a full subcategory of (Gr). Note that (Top) is not a subcategory of (Ens), since a set can have different topologies. Similarly, (Banach) is not a subcategory of (Ab).

Exercises:

1. Show that the following holds for a universe $\underline{U}$:
   a) If $X \in \underline{U}$, then $\{X\} \in \underline{U}$.
   b) If $X,Y \in \underline{U}$, then $(X,Y) \in \underline{U}$.
   c) If $(X_i)_I$ is a family of sets with $I \in \underline{U}$ and each $X_i \in \underline{U}$, then $\prod_I X_i \in \underline{U}$.
   d) If $X \in \underline{U}$, then card $X <$ card $\underline{U}$.
2. Verify that $\underline{U}$-(Ens) is a $\underline{U}$-category which is not $\underline{U}$-small.
3. Let $\underline{C}$ be any category. Show that $Hom(A,A)$ is a semigroup with identity, for each object A.
4. Let $\underline{C}$ be any category. For each pair A,B of objects, write $A \leq B$ if $Hom(A,B) \neq \emptyset$. Show that this defines a preordering on $Ob(\underline{C})$.

§ 2. Some special morphisms and objects.

Let $\underline{C}$ be a category. A morphism $\alpha: A \to B$ is an isomorphism if there exists $\beta: B \to A$ such that $\beta\alpha = 1_A$, $\alpha\beta = 1_B$. A and B are then isomorphic objects. Isomorphy is clearly an equivalence relation on the set $Ob(\underline{C})$. If β is only a right inverse of α, i.e. $\alpha\beta = 1_B$, then

α is called a retraction, while β is a coretraction. (Note that "coretraction" is the dual notion to "retraction", since α is a retraction in $\underline{C}^o$ if and only if α is a coretraction in $\underline{C}$).

Next we want to extend the concepts of injectiveness and surjectiveness of functions, well-known in e.g. the categories (Ens) and (Gr), to general categories. It turns out that the following definitions are the proper ones for this purpose:

Definition. $\alpha: A \to B$ is a monomorphism if for every object C and every pair of morphisms $\xi, \eta: C \to A$ with $\alpha\xi = \alpha\eta$, one has $\xi = \eta$. Dually, α is an epimorphism if for every object C and every pair of morphisms $\xi, \eta: B \to C$ with $\xi\alpha = \eta\alpha$, one has $\xi = \eta$.

Examples:

1. In the categories (Ens), (Top) and (Ab) one has monomorphism = injection, epimorphism = surjection. Let us e.g. verify this in the case of abelian groups. It is quite clear that every injection must be a monomorphism and that every surjection is an epimorphism. Now suppose $\alpha: A \to B$ is a monomorphism between abelian groups. If $x, y \in A$ and $\alpha(x) = \alpha(y)$, define two homomorphisms $\xi, \eta: \underline{Z} \to A$ ($\underline{Z}$ stands for the integers) by $\xi(n) = nx$, $\eta(n) = ny$. Then $\alpha\xi = \alpha\eta$, so $\xi = \eta$, i.e. $x = y$. Next we suppose that α is an epimorphism. Define two homomorphisms $\xi, \eta: B \to B/\mathrm{Im}\,\alpha$ as ξ = natural map, η = zero map. Then $\xi\alpha = \eta\alpha$, so $\xi = \eta$, i.e. $\mathrm{Im}\,\alpha = B$, and α is surjective.
2. In (Gr) it is easy to see that monomorphism = injection, and that every surjection is an epimorphism. It may be shown that every epimorphism $\alpha: A \to B$ is surjective, but this is a little complicated because $\mathrm{Im}\,\alpha$ is not necessarily a normal subgroup of B, so we cannot always form $B/\mathrm{Im}\,\alpha$ as above. We refer to [7] or [18] for a proof.
3. Every retraction is an epimorphism, and every coretraction is a monomorphism. In particular, every isomorphism is both an epimorphism and a monomorphism. The category is called balanced if the converse also holds, i.e. if every morphism which is both an epimorphism and a monomorphism also is an isomorphism. (Ens) and (Gr) are balanced, while (Top) is not balanced.

If $\alpha: A \to B$ is a monomorphism, we call A a <u>subobject</u> of B (this is actually an abuse of language, since subobjects should be considered as equivalence classes of monomorphisms, with the equivalence defined below; this would however lead to complicated notations). If also $\beta: A' \to B$ is a monomorphism, then we write $A \subset A'$ if there exists a morphism $\gamma: A \to A'$ such that $\beta\gamma = \alpha$ (note that γ will then be a monomorphism). α and β are called <u>equivalent</u> if γ is an isomorphism (i.e. if $A \subset A'$ and $A' \subset A$). Dually we define <u>quotient objects</u>.

An object 0 is called <u>initial</u> if $\mathrm{Hom}(0,A)$ consists of exactly one element for each object A. Any two initial objects in $\underline{C}$ are isomorphic, as is easily verified. Dually we call 0 <u>final</u> if $\mathrm{Hom}(A,0)$ consists of exactly one element for each object A. An object which is both initial and final is called a <u>zero object</u>. If there exists a zero object 0 in $\underline{C}$, then we may define a <u>zero morphism</u> $o_{A,B}$ (or simply o) for each pair A,B of objects: $o_{A,B}$ is the composition $A \to 0 \to B$. Note that this morphism is independent of the particular choice of zero object 0.

Examples:

3. (Ens) and (Top) have the initial object $\emptyset$, while every set (space) consisting of one element is final.
4. In $(\mathrm{Ens})_o$ and $(\mathrm{Top})_o$, every pair $(\{x\}, x)$ is a zero object.
5. (Gr) and (Ab) have zero objects.

<u>Exercises:</u>

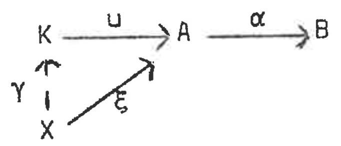
1. Let $\alpha: A \to B$ and $\beta: B \to C$ be morphisms. Show that if α and β are monomorphisms, then so is $\beta\alpha$. Also show that if $\beta\alpha$ is a monomorphsim, then so is α. Dualize.
2. Show that if a morphism α is both a retraction and a monomorphism, then α is an isomorphism.
3. Suppose $\alpha: A \to B$ is a morphism such that for each object C, the obvious function $\mathrm{Hom}(C,A) \to \mathrm{Hom}(C,B)$ is a bijection. Show that α is an isomorphism.
4. Show that in the category of Hausdorff spaces (and continuous maps), $f: X \to Y$ is an epimorphism if $f(X)$ is dense in Y. Also show that if f is an epimorphism and Y is regular, then $f(X)$ is dense in Y.
5. Show that the category of compact Hausdorff spaces is balanced (cf. [2], § 8.3, 9.4).

§ 3. Kernels and cokernels.

Let $\underline{C}$ be a category with a zero object 0.

Definition. A *kernel* of $\alpha: A \to B$ is a morphism $u: K \to A$ such that

1) $\alpha u = 0$,
2) for every morphism $\xi: X \to A$ such that $\alpha\xi = 0$, there exists a unique $\gamma: X \to K$ such that $\xi = u\gamma$.



Proposition 1. i) Every kernel is a monomorphism.
ii) Any two kernels of α are equivalent subobjects.

Proof. Let $\alpha: A \to B$ have the kernel $u: K \to A$.

i) If there are morphisms $\beta, \beta': X \to K$ such that $u\beta = u\beta'$, then let $\xi = u\beta = u\beta'$. Since $\alpha\xi = 0$, ξ should have a unique factorization over u. Hence $\beta = \beta'$.

ii) Suppose also $u': K' \to A$ is a kernel of α. Since $\alpha u' = 0$, there exists $\gamma: K' \to K$ with $u\gamma = u'$. Since $\alpha u = 0$, there also exists $\beta: K \to K'$ with $u'\beta = u$. Hence

$$u\gamma\beta = u'\beta = u = u \cdot 1_K$$

$$u'\beta\gamma = u\gamma = u' = u' \cdot 1_{K'}.$$

But u and u' are monomorphsims, so $\gamma\beta = 1_K$ and $\beta\gamma = 1_{K'}$, i.e. β is an isomorphism.

Because of ii) we may speak of *the* kernel of α (with the same abuse of language as when speaking about subobjects). We will write $u = \ker \alpha$, $K = \operatorname{Ker} \alpha$ when $u: K \to A$ is a kernel of α. If the kernel exists for any morphism in $\underline{C}$, then we say that "$\underline{C}$ has kernels".

Proposition 2. If $\alpha: A \to B$ is a monomorphism, then $\operatorname{Ker} \alpha = 0$.

Proof. If $\xi: X \to A$ is such that $\alpha\xi = o$, then $\alpha\xi = \alpha\, o_{X,A}$. But α is a monomorphism, so $\xi = o_{X,A}$, i.e. ξ may be factored over 0.

Dualizing the definition of kernels we obtain cokernels; the explicit definition is left to the reader. Note that every cokernel is an epimorphism and that the cokernel of an epimorphism is zero. We use the notations coker α, resp. Coker α, for the cokernel of α.

Examples:

1. In (Ab): let $\alpha: A \to B$ be a group homomorphism. Then it is easily verified that
   $\operatorname{Ker} \alpha = \{x \in A \mid \alpha(x) = 0\}$,
   $\operatorname{Coker} \alpha = B / \operatorname{Im} \alpha$.
2. In (Gr): let $\alpha: G \to H$ be a group homomorphism. Then
   $\operatorname{Ker} \alpha = \{x \in G \mid \alpha(x) = 1\}$
   $\operatorname{Coker} \alpha = H / \overline{\operatorname{Im} \alpha}$
   where $\overline{\operatorname{Im} \alpha}$ denotes the unique smallest normal subgroup of H containing $\operatorname{Im} \alpha$.
3. In $(Ens)_o$: let $\alpha: (X, x_o) \to (Y, y_o)$ be a morphism. Then
   $\operatorname{Ker} \alpha = (\{x \mid \alpha(x) = y_o\}, x_o)$
   $\operatorname{Coker} \alpha = (Y / \operatorname{Im} \alpha, \overline{y_o})$
   where $Y / \operatorname{Im} \alpha$ is obtained from Y by identifying all elements im $\operatorname{Im} \alpha$. Note that α may have $\operatorname{Ker} \alpha = o$ without being a monomorphism, but that $\operatorname{Coker} \alpha = o$ if and only if α is an epimorphism.

In categories without zero objects one may sometimes use "equalizers" as subtitutes for kernels.

Definition. Let $\alpha, \alpha': A \to B$ be a pair of morphisms. An equalizer for (α, α') is a morphism $u: K \to A$ such that

1) $\alpha u = \alpha' u$,
2) for every morphism $\xi: X \to A$ such that $\alpha\xi = \alpha'\xi$, there exists a unique $\gamma : X \to K$ such that $\xi = u\gamma$.

As in the case of kernels one shows that u must be a monomorphism and that any two equalizers for (α,α') must be equivalent. We write $K = \text{Equ}(\alpha,\alpha')$. There is a dual notion of coequalizer, denoted by $\text{Coequ}(\alpha,\alpha')$.

Example:

4. In (Ens), (Gr) etc, we have $\text{Equ}(\alpha,\alpha') = \{x \mid \alpha(x) = \alpha'(x)\}$.

Exercises:

Let $\underline{C}$ be a category with zero object.

1. Show that $\ker(\text{coker}(\ker \alpha)) = \ker \alpha$ whenever the involved ker and coker exist.
2. Show that if every monomorphism in $\underline{C}$ is a kernel of some morphism, then $\underline{C}$ is balanced (use ex. 1).
3. Show that the full subcategory of (Ab) consisting of the torsion groups has kernels and cokernels.

§ 4. Products and coproducts.

Let $\underline{C}$ be any category.

Definition. Let $(A_i)_{i\in I}$ be a family of objects in $\underline{C}$. A product of $(A_i)_I$ is an object A together with morphisms $p_i\colon A \to A_i$ $(i\in I)$, such that for each object X and morphisms $\alpha_i\colon X \to A_i$ $(i\in I)$ there is a unique morphism $\alpha\colon X \to A$ with $p_i\,\alpha = \alpha_i$ $(i\in I)$.

To prove the uniqueness of the product (up to isomorphisms), let $(B,\ q_i\colon B \to A_i)$ be another product of $(A_i)_I$. Since A is a product we have $\alpha\colon B \to A$ with $p_i\alpha = q_i$, and since B is a product we have $\beta\colon A \to B$ with $q_i\beta = p_i$. Then

$$p_i\alpha\beta = q_i\beta = p_i = p_i 1_A \Rightarrow \alpha\beta = 1_A \text{ by unicity;}$$

$$q_i\beta\alpha = p_i\alpha = q_i = q_i 1_B \Rightarrow \beta\alpha = 1_B \quad \text{"} \quad .$$

Hence A and B are isomorphic and, furthermore, the isomorphism α has the property that $p_i\alpha = q_i$.

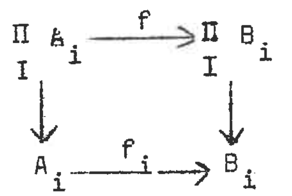
We denote the product object by $\prod_I A_i$ and call the morphisms p_i projections. For finite index sets I one usually writes $A_1 \times \ldots \times A_n$ for the product. If the product exists for all finite families, then $\underline{C}$ is said to "have finite products", and if the product exists for all families indexed by a set belonging to a certain universe $\underline{U}$, then $\underline{C}$ "has $\underline{U}$-products".

It does not follow in general from the definition of products that the projections p_i are epimorphisms. This may however be proved if we assume that $\mathrm{Hom}(A,B) \neq \emptyset$ for every pair of objects A,B. For given a family $(A_i)_I$ of objects, we may then for each fixed $j \in I$ define morphisms $f_i \colon A_j \to A_i$ as

$$\begin{cases} f_j = 1_{A_j} \\ f_i = \text{arbitrary choice, } i \neq j. \end{cases}$$

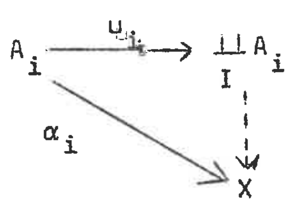
By the definition of product there exists $u_j \colon A_j \to \prod A_i$ such that $p_j u_j = 1$. So p_j is a retraction, hence in particular an epimorphism. If $\underline{C}$ has a zero object, then we may choose f_i above to be zero when $i \neq j$. The resulting monomorphisms u_j for this particular choice are called injections.

Let us now return to a general category $\underline{C}$ and suppose we have a family of morphisms $f_i \colon A_i \to B_i$ $(i \in I)$. Then it is clear that there exists a unique morphism $f\colon \prod_I A_i \to \prod_I B_i$ (assuming the existence of these products) making the following diagram commutative:



Write $f = \prod_I f_i$.

The dual notion to product is coproduct (or direct sum). It is denoted either as $\bigoplus_I A_i$ or as $\coprod_I A_i$. The explicit definition is left to the reader and we just indicate it by the diagram



.

When all the objects A_i are equal to some object A, we denote the coproduct by A^I.

If $\underline{C}$ has a zero object, then (dualizing the above results) the canonical injections u_i are monomorphisms and one may define projections $p_j: \coprod_I A_i \to A_j$ such that

$$p_j u_i = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Examples:

1. $\underline{U}$-(Ens) has both $\underline{U}$-products and $\underline{U}$-coproducts. The product is just the ordinary cartesian product, while the coproduct is the disjoint union. Similarly for $\underline{U}$-(Top).
2. In $(\text{Ens})_0$, the product is essentially the cartesian product, while the coproduct is obtained by taking the disjoint union with all base points identified.
3. In (Ab), the product is the ordinary direct product (i.e. the cartesian product with group operations defined component-wise). The coproduct coincides with the direct sum (i.e. the subgroup of the direct product which consists of elements with all but a finite number of coordinates equal to zero).
4. In (Gr), the product coincides with the direct product, while the coproduct is the same as the free product (see [16], Ch. I, Prop. 8).

Let $(A_i)_I$ and $(B_j)_J$ be two families of objects in $\underline{C}$. Every morphism

$$\coprod_I A_i \to \prod_J B_j$$

defines in a natural way morphisms $A_i \to B_j$ for each pair (i,j). Conversely, given a morphism $\alpha_{ji}: A_i \to B_j$ for each pair (i,j), one sees by using the definitions of products and coproducts, that there exists a unique morphism α such that $p_j\, \alpha\, u_i = \alpha_{ji}$, all (i,j).

$$\begin{array}{ccc} A_i & \xrightarrow{\alpha_{ji}} & B_j \\ {\scriptstyle u_i}\downarrow & & \uparrow{\scriptstyle p_j} \\ \coprod_I A_i & \xrightarrow{\alpha} & \prod_J B_j \end{array}$$

Consider in particular the case when $\underline{C}$ has zero objects and $\delta_{ji}: A_i \to A_j$ is defined as

$$\delta_{ji} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

We then obtain a canonical morphism $\delta: \coprod_I A_i \to \prod_I A_i$, which in general is neither a monomorphism nor an epimorphism.

Exercises:

1. Let $\underline{C}$ have zero objects and consider a product $A_1 \times A_2$ with projections p_1, p_2 and injections u_1, u_2. Show that $u_1 = \ker p_2$.
2. Show that the products and coproducts in the $\underline{U}$-category $\underline{C}$ may be characterized by the formulas

$$\mathrm{Hom}\,(X, \prod_I Y_i) \cong \prod_I \mathrm{Hom}\,(X, Y_i)$$

$$\mathrm{Hom}\,(\coprod_I X_i, Y) \cong \prod_I \mathrm{Hom}\,(X_i, Y)$$

   where $I \in \underline{U}$ and the second member products are taken in $\underline{U}$-(Ens).
3. Show that the category of abelian torsion groups has coproducts but not infinite products.

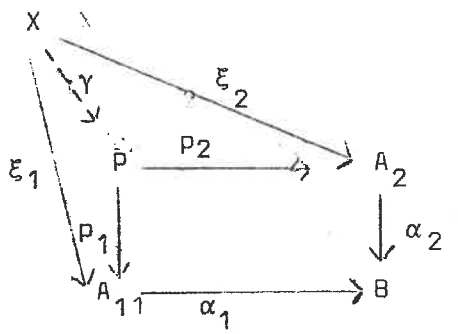
§ 5. Pullbacks and pushouts.

Let $\underline{C}$ be a category.

Definition. Let $\alpha_1: A_1 \to B$ and $\alpha_2: A_2 \to B$ be morphisms in $\underline{C}$. A pullback for α_1 and α_2 is an object P together with morphisms $p_1: P \to A_1$, $p_2: P \to A_2$ such that:

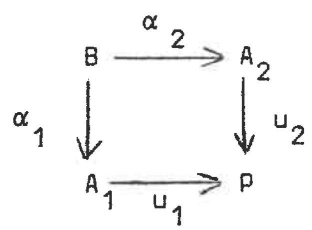
1) $\alpha_1 p_1 = \alpha_2 p_2$;

2) for every object X and morphisms $\xi_1: X \to A_1$, $\xi_2: X \to A_2$ such that $\alpha_1 \xi_1 = \alpha_2 \xi_2$, there exists a unique $\gamma: X \to P$ such that $p_1 \gamma = \xi_1$ and $p_2 \gamma = \xi_2$.

(*)


It is easily verified that the pullback is unique up to isomorphism. We will denote it as $P = A_1 \times_B A_2$, even though this notation does not record the given morphisms α_1, α_2.

Dualizing the definition of pullbacks we obtain pushouts. A pushout diagram for $\alpha_1 : B \to A_1$ and $\alpha_2 : B \to A_2$ looks like



We write $A_1 \sqcup_B A_2$ for the pushout.

Proposition 3. If α_2 in the pullback diagram $(*)$ is a monomorphism, then also p_1 is a monomorphism.

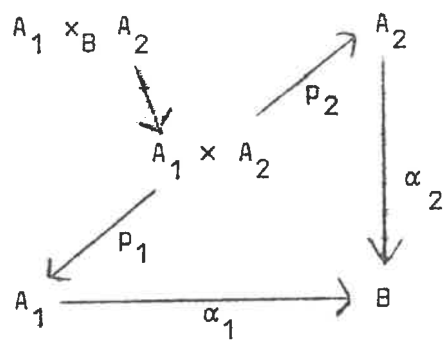
Proof. Suppose we have $\xi, \eta : X \to P$ such that $p_1\xi = p_1\eta$. Then $\alpha_1 p_1 \xi = \alpha_1 p_1 \eta$ and by commutativity $\alpha_2 p_2 \xi = \alpha_2 p_2 \eta$. But α_2 is a monomorphism, so $p_2\xi = p_2\eta$. Now we have a pair of morphisms $p_1\xi : X \to A_1$ and $p_2\xi : X \to A_2$ which have the two factorizations ξ and η through P. By definition of pullback we must then have $\xi = \eta$.

In the situation described by the proposition, we call the pullback P an inverse image of A_2 by α_1 and denote it as $\alpha_1^{-1}(A_2)$.

If both α_1 and α_2 are monomorphisms, then their pullback P is a subobject of both A_1 and A_2, and it is reasonable to call it the intersection of A_1 and A_2, denoted as $A_1 \cap A_2$.

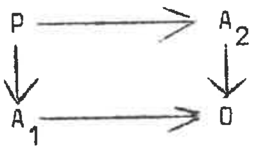
Proposition 4. If $\underline{C}$ has equalizers and finite products, then it has pullbacks.

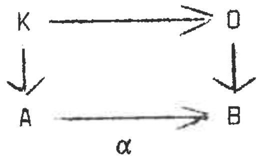
Proof. Let $\alpha_1 : A_1 \to B$ and $\alpha_2 : A_2 \to B$ be given morphisms. Consider the product $A_1 \times A_2$ with its projections p_1 and p_2 to A_1 resp. A_2. We assert that $A_1 \times_B A_2 = \mathrm{Equ}\,(\alpha_1 p_1, \alpha_2 p_2)$, and leave to the reader to verify that this equalizer has the universal properties that the pullback should have.



Examples:

1. It follows from the construction made in the proof of Prop. 4 that in the categories (Ens), (Gr) and (Ab) we have $A_1 \times_B A_2 = \{(x,y) \in A_1 \times A_2 \mid \alpha_1(x) = \alpha_2(y)\}$. Inverse images and intersections have their usual meaning.
2. Suppose $\underline{C}$ has a final object 0. Then $A_1 \times_0 A_2 = A_1 \times A_2$.

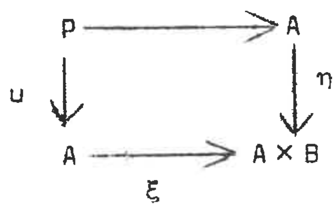




3. Suppose $\underline{C}$ has a zero object 0 and let $\alpha: A \to B$ be a morphism. Then $\operatorname{Ker} \alpha = A \times_B 0$.

<u>Exercises:</u>

1. Give explicit formulas for the pushout and cointersection in (Ab).
2. Show that if $\underline{C}$ has pullbacks and finite intersections then it has equalizers (Hint: the equalizer of $\alpha, \beta: A \to B$ may be obtained as u in the pullback diagram



if ξ and η are suitably defined).

3. Suppose $\underline{C}$ has a zero object and $A \amalg B$ exists. Show that if A and B are considered as subobjects of $A \amalg B$, then $A \cap B = 0$.

§ 6. Preadditive categories.

Thus far we have worked with categories of a quite general nature, but in this § we will impose some more algebraic structure on our categories.

Definition. A category $\underline{C}$ is preadditive if each set $\text{Hom}(A,B)$ is an abelian group and the composition mappings $\text{Hom}(B,C) \times \text{Hom}(A,B) \to$ $\to \text{Hom}(A,C)$ are bilinear.

If $\underline{C}$ is preadditive, then in particular each set $\text{Hom}(A,B)$ has an element $o'_{A,B}$ which is a zero element for the group structure.

If now $\underline{C}$ also has a zero element 0, then there are zero morphisms $o_{A,B}$ in the sense of § 2. We want to show that $o'_{A,B}$ and $o_{A,B}$ coincide. First note that by bilinearity we have for any morphism $\alpha: A \to B$ that $o'_{B,C} \cdot \alpha = o'_{A,C}$ and $\alpha \cdot o'_{C,A} = o'_{C,B}$. So in particular we have $o'_{0,B} \cdot o'_{A,0} = o'_{A,B}$ and thus $o'_{A,B}$ may be factored over 0, i.e. $o'_{A,B} = o_{A,B}$. In the sequel we will denote the zero element of the group $\text{Hom}(A,B)$ by $o_{A,B}$ or o, even when $\underline{C}$ has no zero object.

Note that if $\underline{C}$ is preadditive, then the following holds for a morphism α:

$$\alpha \text{ is a monomorphism if and only if } \alpha\xi = o \Rightarrow \xi = o;$$

$$\alpha \text{ is an epimorphism if and only if } \xi\alpha = o \Rightarrow \xi = o.$$

Examples:

1. (Ab) and (Banach) are preadditive categories.
2. If A is a ring, then the categories $\text{Mod.}(A)$ of left A-modules and $\text{Mod}(A).$ of right A-modules are preadditive.
3. Recall from § 1 that a category with only one identity is the same as a semigroup with identity. Preadditivity of such a category means that there is a second binary operation $(\alpha,\beta) \to \alpha+\beta$,

under which it is an abelian group, such that

$$\alpha(\beta + \beta') = \alpha\beta + \alpha\beta', \quad (\alpha + \alpha')\beta = \alpha\beta + \alpha'\beta.$$

So a preadditive category with only one identity is the same as a ring with identity. Its dual category is commonly called the <u>opposite ring</u>.

<u>Definition</u>. A preadditive category is called <u>additive</u> if it has a zero object and has finite products and coproducts.

Let $\underline{C}$ be additive and consider a finite product $\prod_1^n A_i$. We have previously defined

$$\begin{array}{ll} \text{projections} & p_j : \Pi A_i \to A_j \\ \text{injections} & u_j : A_j \to \Pi A_i \end{array} \quad \text{with} \quad p_j u_i = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

We also have $\sum_1^n u_i p_i = 1.$

Proof: $p_j(\Sigma u_i p_i) = \Sigma p_j u_i p_i = p_j = p_j \cdot 1$, so by the definition of products we have $\Sigma u_i p_i = 1.$

Dually we have for the coproduct $\bigoplus_1^n A_i$:

$$\begin{array}{ll} \text{injections} & u'_j : A_j \to \bigoplus A_i \\ \text{projections} & p'_j : \bigoplus A_i \to A_j \end{array} \quad \text{with} \quad p'_j u'_i = \delta_{ij} \quad \text{and} \quad \sum_1^n u'_i p'_i = 1.$$

There is a very useful converse of these results:

<u>Proposition 5</u>. Let $\underline{C}$ be a preadditive category. Suppose there are given morphisms $u_i : A_i \to A$ and $p_i : A \to A_i$ $(i = 1,\ldots,n)$ such that $p_j u_i = \delta_{ij}$ and $\Sigma u_i p_i = 1$. Then A is a product (and by symmetry, also a coproduct) of the objects A_i, with injections u_i and projections p_i.

<u>Proof</u>. Suppose there are given morphisms $\alpha_i : X \to A_i$. Define $\alpha : X \to A$ as $\alpha = \Sigma u_i \alpha_i$. Then $p_j \alpha = \sum_i p_j u_i \alpha_i = \alpha_j$. There is only one choice for α, since if $\alpha' : X \to A$ is such that $p_i \alpha' = \alpha_i$, then $\alpha' = \Sigma u_i p_i \alpha' = \Sigma u_i \alpha_i = \alpha$.

Hence the notions of product and of coproduct "coincide" for preadditive categories. We will now prove a useful formula for composition of morphisms between finite products. Let $\underline{C}$ be additive and consider

$$\prod_K A_K \xrightarrow{\alpha} \prod_J B_j \xrightarrow{\beta} \prod_I C_i$$

where I, J and K are finite. α and β are represented by matrices (α_{jk}) and (β_{ij}) with

$$\alpha_{jk} = p_j^B \alpha u_k^A \quad , \quad \beta_{ij} = p_i^C \beta u_j^B.$$

$\beta\,\alpha$ is represented by a matrix with (i,k)-entry

$$p_i^C \beta \alpha u_k^A = p_i^C \beta \left(\sum_j u_j^B p_j^B\right) \alpha u_k^A = \sum_j \beta_{ij} \alpha_{jk},$$

i.e. the matrix representing $\beta\,\alpha$ is the product of the matrices representing β and α.

Application:
Let $\alpha,\beta : A \to B$ be morphisms. Matrix multiplications then give $\alpha + \beta$ as the composition

$$A \xrightarrow{\begin{pmatrix}1\\1\end{pmatrix}} A \oplus A \xrightarrow{\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}} B \oplus B \xrightarrow{(1,1)} B.$$

We finally note that if $\underline{C}$ is additive, then:

1) α is a monomorphism $\Leftrightarrow$ $\operatorname{Ker}\alpha = 0$,
   α is an epimorphism $\Leftrightarrow$ $\operatorname{Coker}\alpha = 0$.
2) $\operatorname{Equ}(\alpha,\beta) = \operatorname{Ker}(\alpha - \beta)$,
   $\operatorname{Ker}\alpha = \operatorname{Equ}(\alpha, 0)$.
3) $\underline{C}$ has kernels if and only if $\underline{C}$ has pullbacks (Prop. 4).

Exercises:

Let $\underline{C}$ be a preadditive category.

1. Let A and B be objects in $\underline{C}$. Show that:
   a) Hom (A,A) is a ring under composition and addition.

b) Hom (A,B) is naturally a right module over Hom(A,A) and a left module over Hom (B,B).

2. Suppose that the two morphisms $\alpha_1: A_1 \to B$ and $\alpha_2: A_2 \to B$ have a pullback. Show that $A_1 \times_B A_2 \to A_2$ is a monomorphism if and only if α_1 is a monomorphism.

§ 7. Abelian categories.

Roughly stated, an abelian category is an additive category in which one can handle exact sequences in about the same manner as one is used to in module categories.

Suppose $\underline{C}$ is a category which has kernels and cokernels. Any morphism α has a factorization as indicated by the commutative diagram

$$\begin{array}{ccc} \text{Ker } \alpha & & \text{Coker } \alpha \\ \downarrow & & \uparrow \\ A & \xrightarrow{\alpha} & B \\ \lambda \downarrow & & \uparrow \mu \\ \text{Coker (ker } \alpha) & \xrightarrow[\bar{\alpha}]{} & \text{Ker (coker } \alpha) \end{array}$$

where the existence of $\bar{\alpha}$ is obtained as follows:

coker $\alpha \cdot \alpha = 0 \Rightarrow \alpha = \mu\beta$ for some $\beta: A \to$ Ker (coker α). Then $\mu\beta \cdot$ ker $\alpha = \alpha \cdot$ ker $\alpha = 0$, so $\beta \cdot$ ker $\alpha = 0$ since μ is a monomorphism. Hence β may be factored as $\beta = \bar{\alpha}\lambda$. Also note that $\bar{\alpha}$ is uniquely determined by α.

Consider as an example the particular case of $\underline{C} = (Ab)$. There we have that Coker (ker α) = A/Ker α and Ker (coker α) = Im α, and consequently $\bar{\alpha}$ is an isomorphism. The following definition is therefore reasonable:

Definition. A category $\underline{C}$ is abelian if

AB 1: $\underline{C}$ is additive.

AB 2: $\underline{C}$ has kernels and cokernels.

AB 3: $\bar{\alpha}$: Coker (ker α) $\to$ Ker (coker α) is an isomorphism, for every morphism α.

This definition is self-dual, i.e. $\underline{C}$ is abelian if and only if $\underline{C}^o$ is. For every morphism α in an abelian category we define its image as $\operatorname{Im} \alpha = \operatorname{Ker} (\operatorname{coker} \alpha)$. The dual notion of coimage is rather superfluous for abelian categories, since image and coimage are canonically isomorphic.

Some examples of abelian categories:

1. (Ab) is abelian.
2. If A is a ring, then $\operatorname{Mod}_{\ell}(A)$ and $\operatorname{Mod}(A)_r$ are abelian.
3. If $\underline{O}$ is a sheaf of rings over a topological space, then the category of $\underline{O}$-Modules is abelian ([11], p. 131).
4. The category of commutative algebraic group schemes of finite type over a field is an abelian category.

For the rest of this § we will assume that $\underline{C}$ is abelian.

Proposition 6. Every abelian category is balanced.

Proof. Clear from AB 3.

Proposition 7. If α is a monomorphism, then $\alpha = \ker (\operatorname{coker} \alpha)$.
If α is an epimorphism, then $\alpha = \operatorname{coker} (\ker \alpha)$.

Proof. This is also quite clear from AB 3.

For every morphism α there is a canonical factorization $A \xrightarrow{\alpha'} \operatorname{Im} \alpha \xrightarrow{u} B$ where α' is an epimorphism and u is a monomorphism. We will show that the image is universal with respect to this property (and thus deserves its name).

Proposition 8. If $\alpha: A \to B$ has a factorization $A \xrightarrow{\xi} I \xrightarrow{\eta} B$ where η is a monomorphism, then $\operatorname{Im} \alpha \subset I$. If also ξ is an epimorphism, then $\operatorname{Im} \alpha$ and I are equivalent subobjects of B.

Proof. Suppose η is a monomorphism. Since $\operatorname{coker} \eta \cdot \alpha = o$, there exists $\gamma: \operatorname{Coker} \alpha \to \operatorname{Coker} \eta$ such that $\operatorname{coker} \eta = \gamma \cdot \operatorname{coker} \alpha$. But then clearly $\operatorname{coker} \eta \cdot \ker (\operatorname{coker} \alpha) = o$, so $\ker (\operatorname{coker} \alpha)$ may be factored over $\ker (\operatorname{coker} \eta) = \eta$ (Prop. 7). Hence $\operatorname{Im} \alpha \subset I$. Let $\varphi: \operatorname{Im} \alpha \to I$ be the factorization.

Now also suppose that ξ is an epimorphism. We have $\eta \varphi \alpha' = \alpha = \eta \xi$, so $\varphi \alpha' = \xi \cdot \varphi$ is therefore an epimorphism, and since it clearly is a monomorphism, it is an isomorphism (Prop. 6).

In passing we mention the following result:

Theorem (Freyd) A category is abelian if and only if

i) it has zero object, pullbacks and pushouts;

ii) every monomorphism is a kernel and every epimorphism is a cokernel.

The point of this is that it is not necessary to assume preadditivity; instead this is forced upon the category by way of e.g. the addition formula at the end of the preceding §. The reader is refered to [9] or [18] for a proof.

We will now introduce the machinery of exact sequences into the abelian category $\underline{C}$.

Definition. A sequence $\ldots\ldots \to A_{i-1} \overset{\alpha_{i-1}}{\to} A_i \overset{\alpha_i}{\to} A_{i+1} \to \ldots$ is called exact at A_i if $\operatorname{Im} \alpha_{i-1} = \operatorname{Ker} \alpha_i$ (as subobjects of A_i). The sequence is exact if it is exact at each A_i.

Proposition 9. $A \overset{\lambda}{\to} B \overset{\mu}{\to} C$ is exact in $\underline{C}$ if and only if $C \overset{\mu^o}{\to} B \overset{\lambda^o}{\to} A$ is exact in $\underline{C}^o$.

Proof. (We have here written λ^o for λ as a morphism in $\underline{C}^o$). Exactness in $\underline{C}$ means that $\operatorname{Ker}(\operatorname{coker} \lambda) = \operatorname{Ker} \mu$, while exactness in $\underline{C}^o$ means that $\operatorname{Ker}(\operatorname{coker} \mu^o) = \operatorname{Ker} \lambda^o$, i.e. $\operatorname{Coker}(\ker \mu) =$ $= \operatorname{Coker} \lambda$ in $\underline{C}$. These two conditions are equivalent (use § 3, exercise 1).

Example:

$0 \to A \overset{\alpha}{\to} B$ is exact if and only if α is a monomorphism.

$A \overset{\alpha}{\to} B \to 0$ is exact if and only if α is an epimorphism.

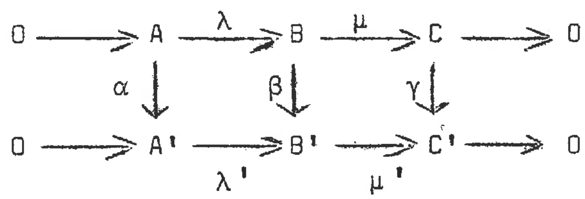
An exact sequence of the form $0 \to A \overset{\lambda}{\to} B \overset{\mu}{\to} C \to 0$ is called a short exact sequence (abbreviated s.e.s.). Exactness here means:

1) λ is a monomorphism and μ is an epimorphism;

2) $\operatorname{Im} \lambda = \operatorname{Ker} \mu \Leftrightarrow \lambda = \ker \mu \Leftrightarrow \mu = \operatorname{coker} \lambda$.

One often writes B/A for C.

Proposition 10. (Short 5 lemma) Suppose the diagram



is commutative with exact rows. Then:

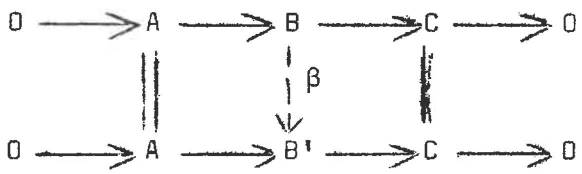
i) If α and γ are monomorphisms, then β is a monomorphism.

ii) Similarly for epimorphisms.

iii) Similarly for isomorphisms.

Proof. Consider i). If $\xi : X \to B$ is given such that $\beta\xi = o$, then $\gamma\mu\xi = \mu'\beta\xi = o$. But γ is a monomorphism, so $\mu\xi = o$. ξ therefore factors over $\lambda = \ker \mu$ as $\xi = \lambda\xi'$. Then $\lambda'\alpha\xi' = \beta\lambda\xi' = \beta\xi = o$, and since $\lambda'\alpha$ is a monomorphism we conclude that $\xi' = o$, and $\xi = o$. Consequently β is a monomorphism. ii) now follows by duality, while iii) is obtained by combining i) and ii).

We will introduce a useful equivalence relation on the set of s.e.s. between two given objects A and C:



The two sequences of this diagram are called equivalent if there exists $\beta: B \to B'$ such that the diagram commutes. If β exists, then it is an isomorphism by Prop. 10, so this really is an equivalence relation.

Proposition 11. The following properties of a s.e.s. $0 \to A \xrightarrow{\lambda} B \xrightarrow{\mu} C \to 0$ are equivalent:

a) It is equivalent to the s.e.s.
$0 \to A \to A \oplus C \to C \to 0$.

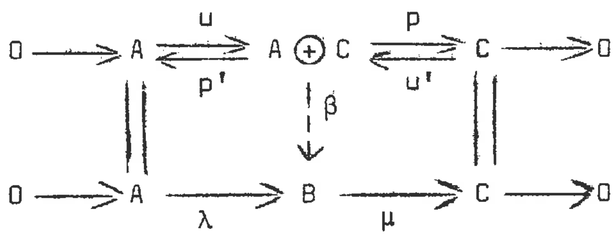
b) μ is a retraction, i.e. there exists $\xi: C \to B$ such that $\mu\xi = 1$.

c) λ is a coretraction, i.e. there exists $\eta: B \to A$ such that $\eta\lambda = 1$.

Definition. A sequence satisfying these conditions is called split.

Proof. It is understood that the morphisms in (a) should be the injection $A \to A \oplus C$ and the projection $A \oplus C \to C$; the exactness is obvious. By duality it suffices to prove the equivalence of (a) and (b).

(a) $\Rightarrow$ (b): If β exists, put $\xi = \beta\mu'$. Then $\mu\xi = \mu\beta u' = pu' = 1$.
(b) $\Rightarrow$ (a): If ξ exists, put $\beta = \lambda p' + \xi p$. Then commutativity holds, for $\beta u = \lambda p' u + \xi pu = \lambda + o = \lambda$ and $\mu\beta = \mu\lambda p' + \mu\xi p = o + p = p$



Exercises:

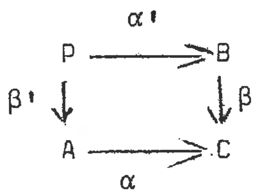
1. Show that the category of abelian torsion groups is abelian.
2. Show that the category of torsion-free abelian groups satisfies AB 1 and AB 2 but not AB 3. (Hint: if $\alpha: A \to B$, then Coker $\alpha = B / \overline{\alpha(A)}$ where $\overline{\alpha(A)}$ is the smallest pure subgroup of B containing $\alpha(A)$; cf. [14], § 7).
3. Show that if $\alpha: A \to B$ is a morphism in an abelian category, then the sequence

$$0 \to \operatorname{Ker} \alpha \to A \xrightarrow{\alpha} B \to \operatorname{Coker} \alpha \to 0$$

   is exact.
4. Given $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ in an abelian category, show that:
   α epimorphism $\Rightarrow$ Coker $\beta\alpha$ = Coker β
   β monomorphism $\Rightarrow$ Ker $\beta\alpha$ = Ker α.

Appendix I.A: Universal epimorphisms.

Let $\underline{C}$ be any category. Consider a pullback diagram

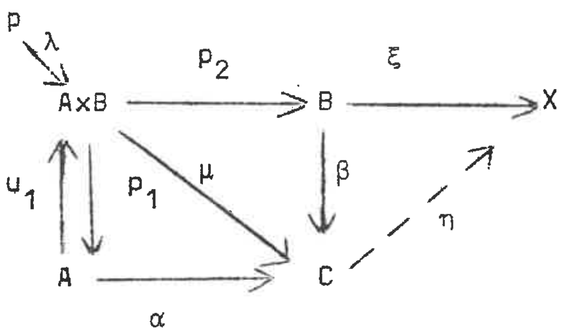


in $\underline{C}$. In § 5 it was proved that if α is a monomorphism, then so is also α'. It is not certain, however, that α is an epimorphism

implies that α' is an epimorphism. We therefore make the following
Definition. An epimorphism $\alpha: A \to C$ is a universal epimorphism if $A x_C B$ exists for any $\beta: B \to C$ and $\alpha': A x_C B \to B$ is an epimorphism.

Important examples of non-universal epimorphisms occur in algebraic geometry (the category of preschemes) and by duality therefore also in the dual of the category of commutative rings (see Appendix I.C for an example).

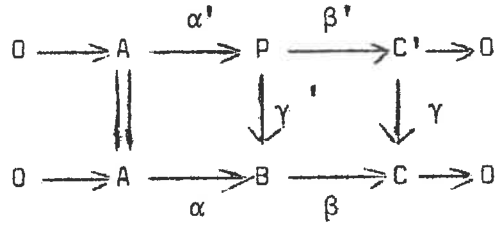
Proposition. In an abelian category all epimorphisms are universal.
Proof. Consider the pullback diagram above with α epimorphic, and suppose $\xi: B \to X$ is such that $\xi\alpha' = o$. We recall from § 5 that $P = \text{Ker}(\alpha p_1 - \beta p_2)$.



$\mu = \alpha p_1 - \beta p_2$

Also note that the sequence $0 \to P \to AxB \to C \to 0$ is exact. The assumption $o = \xi\alpha' = \xi p_2\lambda$ therefore gives $\xi p_2 = \eta\mu$ for some $\eta: C \to X$. Now $\eta\alpha = \eta(\alpha p_1 - \beta p_2)u_1 = \xi p_2 u_1 = o$, so $\eta = o$ and hence $\xi = o$.

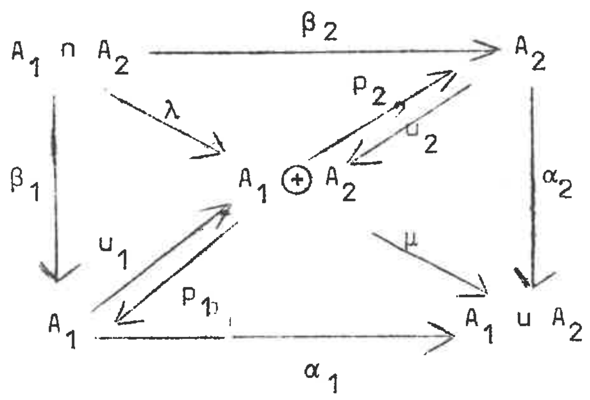
Let $\underline{C}$ be an abelian category. A consequence of the proposition is that we may construct the pullback of a s.e.s. $0 \to A \to B \to C \to 0$ with respect to a morphism $C' \to C$. For this we consider the diagram



where the right square is a pullback, and α' is induced by the two morphisms $\alpha: A \to B$ and [illegible]$: A \to C'$. The left square is then commutative by construction. In particular α' is a monomorphism, and it remains to see that $\alpha' = \ker \beta$. But if $\xi: X \to P$ is such that $\beta'\xi = o$, then $\gamma'\xi$ factors over α as $\gamma'\xi = \alpha\lambda$. $\alpha'\lambda$ and ξ give the same results when going down to B or to C', so $\xi = \alpha'\lambda$ by the property of pullbacks. The upper row of the diagram is thus a s.e.s. which is called the pullback of the lower sequence with respect to γ.

Appendix I.B: Unions in abelian categories.

Let $\underline{C}$ be an abelian category. If $(A_i)_I$ is a family of subobjects of an object A, and if $\bigoplus_I A_i$ exists, then we define the union (or the sum) of the family $(A_i)_I$ as the image of the morphism $\bigoplus_I A_i \to A$ induced by the inclusions $A_i \to A$. It is denoted by $\bigcup_I A_i$ (or $\sum_I A_i$). We will look in more detail at the special case $A_1 \cup A_2 \subset A$.
Consider the diagram



$\lambda = u_1\beta_1 - u_2\beta_2$

with the canonical morphisms.

Proposition 1. The diagram has the following properties:

(i) the sequence

$$0 \to A_1 \cap A_2 \xrightarrow{\lambda} A_1 \oplus A_2 \xrightarrow{\mu} A_1 \cup A_2 \to 0$$

is exact.

(ii) The outer square is both a pullback and a pushout diagram.

Proof. (i): We want to prove that $\lambda = \ker \mu$. First note that by definition we have $\alpha_1 = \mu u_1$, $\alpha_2 = \mu u_2$, and that gives $\mu = \alpha_1 p_1 + \alpha_2 p_2$. Now suppose $\mu \xi = o$ for some $\xi : X \to A_1 \oplus A_2$. Then $\alpha_1 p_1 \xi = -\alpha_2 p_2 \xi$. It is quite clear that the outer square is a pullback, so there exists $\xi' : X \to A_1 \cap A_2$ with $\beta_1 \xi' = p_1 \xi$ and $\beta_2 \xi' = -p_2 \xi$. Then $\lambda \xi' = u_1 \beta_1 \xi' - u_2 \beta_2 \xi = (u_1 p_1 + u_2 p_2)\xi = \xi$ as desired. It is also clear that ξ' is unique.

(ii) It has already been remarked that it is a pullback diagram. The easy verification of the pushout property is left to the reader.

Proposition 2. Let $(A_i)_1^n$ be a finite family of subobjects of A. The canonical morphism $\bigoplus_1^n A_i \to \bigcup_1^n A_i$ is an isomorphism if and only if $A_i \cap (\bigcup_{j \neq i} A_j) = 0$ for each i.

Proof. By induction we are easily reduced to the case $n = 2$. The assertion is then an immediate consequence of Prop. 1 (i).

Proposition 3. (Second Noether isomorphism theorem). Let A_1 and A_2 be subobjects of A. The diagram above induces a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_1 \cap A_2 & \longrightarrow & A_1 & \longrightarrow & A_1/A_1 \cap A_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} & & \Big\Vert{\scriptstyle \alpha''} & & \\
0 & \longrightarrow & A_2 & \longrightarrow & A_1 \cup A_2 & \longrightarrow & A_1 \cup A_2/A_2 & \longrightarrow & 0.
\end{array}$$

Proof. Since α' and α are monomorphisms, the induced morphism $\alpha'' : A_1/A_1 \cap A_2 \to A_1 \cup A_2/A_2$ is a monomorphism (the 5 lemma). To see that α'' is an epimorphism, consider any $\xi : A_1 \cup A_2/A_2 \to X$ such that $\xi \alpha'' = o$. By composition we obtain zero morphisms $A_1 \to X$ and $A_2 \to X$, and since the left square is a pushout, unicity of factorizations implies that $A_1 \cup A_2 \to A_1 \cup A_2/A_2 \to X$ is zero. Hence $\xi = o$.

Appendix I.C: Commutative rings.

Let k be a commutative ring with 1. Let (k-Alg) denote the category of commutative k-algebras with 1, with the k-algebra homomorphisms as morphisms. By taking k = integers we obtain the category of commutative rings as a special case.

It is essential that we include the zero ring $\{0\}$ in (k-Alg), where it becomes a final object. The category also has an initial object, namely the ring k itself.

It is easily verified that the monomorphisms in (k-Alg) are the injective homomorphisms, and that every surjective homomorphism is an epimorphism. But there are lot of epimorphisms which are not surjective. E.g. we have:

Proposition 1. If A is a k-algebra and S is a multiplicatively closed subset of A, then the canonical homomorphism $u: A \to S^{-1}A$ is an epimorphism.

Proof. For the definition of $S^{-1}A$ we refer to [3]. Suppose $\xi,\eta: S^{-1}A \to B$ are morphisms such that $\xi u = \eta u$. For every $\frac{a}{s} \in S^{-1}A$ we then have $\xi(\frac{a}{s}) = \xi(a)\ \xi(s)^{-1} = \eta(a)\ \eta(s)^{-1} = \eta(\frac{a}{s})$, so $\xi = \eta$.

The product in (k-Alg) is just the ordinary direct product. The category also has pullbacks.

Proposition 2. (k-Alg) has pushouts.

Proof. Suppose we are given k-algebra homomorphisms $\beta: A \to B$ and $\gamma: A \to C$. B and C may in a natural way be considered as A-modules, and we form $B \otimes_A C$ which is a k-module. It becomes a k-algebra if multiplication is defined as $(b \otimes c)\cdot(b' \otimes c') = bb' \otimes cc'$. The canonical maps $B \to B \otimes_A C$ and $C \to B \otimes_A C$ are k-algebra homomorphisms. It is now easily verified that the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & B \otimes_A C \end{array}$$

is a pushout diagram in (k-Alg).

In particular we have that the coproduct of the k-algebras B and C is $B \otimes_k C$. Even when B and C are non-zero it may happen that their coproduct is zero. E.g. we may take $k = \underline{Z}$ and $B = \underline{Q}$, $C = \underline{Z}/2\underline{Z}$. This example also shows that the monomorphism $\underline{Z} \to \underline{Q}$ is not a universal monomorphism.

The epimorphisms in (k-Alg) have been studied extensively in later years [20]. Here we will only note the following connection between epimorphisms and coproducts:

Proposition 3. $A \to B$ is an epimorphism in (k-Alg) if and only if in $B \otimes_A B$ one has $1 \otimes b = b \otimes 1$ for every $b \in B$.

Proof. Consider the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta'} \\ B & \xrightarrow[\beta]{} & B \otimes_A B \end{array}$$

where α is the given morphism. If α is an epimorphism, then $\beta = \beta'$. On the other hand, if $\beta = \beta'$ then it follows easily from the pushout property that α is an epimorphism. $\beta = \beta'$ is equivalent to the condition that $1 \otimes b = b \otimes 1$ for all b.

Exercises:

1. Show that $A \to B$ is an epimorphism in (k-Alg) if and only if the map $B \otimes_A B \to B$, given by $b \otimes b' \to bb'$, is an isomorphism.
2. Show that every faithfully flat epimorphism of k-algebras is an isomorphism.

Chapter II. Functors.

§ 1. Basic definitions.

We will now define what might be considered as the morphisms in the "category of categories", namely the functors between categories. If we stick to the original definition of a category as a set with a partially defined binary composition, it is quite clear what a functor should be. Let $\underline{C}$ and $\underline{D}$ be two categories.

Definition. A function $T: \underline{C} \to \underline{D}$ is a functor if

F 1: $T(\beta\alpha) = T(\beta)\, T(\alpha)$ whenever the composition $\beta\alpha$ is defined in $\underline{C}$.

F 2: If e is an identity in $\underline{C}$, then $T(e)$ is an identity in $\underline{D}$.

For every object A of $\underline{C}$ we write $T(A)$ for the object of $\underline{D}$ corresponding to the identity $T(1_A)$; thus $1_{T(A)} = T(1_A)$. If $\alpha: A \to B$ is a morphism in $\underline{C}$, then $T(\alpha) = T(1_B\, \alpha\, 1_A) = T(1_B)\, T(\alpha)\, T(1_A) = = 1_{T(B)}\, T(\alpha)\, 1_{T(A)}$. It follows that $T(\alpha)$ must be a morphism $T(A) \to T(B)$. Hence T induces a function $\mathrm{Hom}_{\underline{C}}(A,B) \to \mathrm{Hom}_{\underline{D}}(T(A), T(B))$ for each pair (A,B).

In practice, a functor $T: \underline{C} \to \underline{D}$ is often obtained the other way round, as a collection of functions:

$\mathrm{Ob}\,(\underline{C}) \to \mathrm{Ob}\,(\underline{D})$,

$\mathrm{Hom}_{\underline{C}}\,(A,B) \to \mathrm{Hom}_{\underline{D}}\,(T(A), T(B))$, all (A,B),

and the axioms F 1, F2 then take the form

F 1': $T(\beta\alpha) = T(\beta)\, T(\alpha)$ when $\beta\alpha$ is defined;

F 2': $T(1_A) = 1_{T(A)}$.

A functor $T: \underline{C} \to \underline{D}$ is called faithful if the induced maps $\mathrm{Hom}\,(A,B) \to \mathrm{Hom}\,(T(A), T(B))$ are injective, and is full if they are surjective. T is an imbedding if it is faithful and takes distinct objects into distinct objects, i.e. the function $T: \underline{C} \to \underline{D}$ is injective.

T is an isomorphism if there is a functor $S: \underline{D} \to \underline{C}$ such that $TS = 1_{\underline{D}}$, $ST = 1_{\underline{C}}$. The notion of isomorphism is however too restrictive, and it is more adequate to consider two categories as "essentially the same" if there is what is called an equivalence between them; T is an equivalence if it is full and faithful and if for every object M in $\underline{D}$ there exists an object A of $\underline{C}$ such that M and $T(A)$ are isomorphic. (It will be found in ch. 3 that this really is an equivalence relation between categories).

When $\underline{C}$ and $\underline{D}$ both are preadditive categories, we call a functor $T: \underline{C} \to \underline{D}$ additive if

F 3: $T(\alpha + \alpha') = T(\alpha) + T(\alpha')$ for $\alpha, \alpha': A \to B$.

In that case T induces a group homomorphism
$Hom(A,B) \to Hom(T(A), T(B))$, all (A,B).

Examples:

1. For each universe $\underline{U}$ we may define a category $\underline{U}$-(Cat) as follows:
   objects: $\underline{U}$-small categories,
   morphisms: functors,
   composition: composition of functors in the obvious way,
   This is clearly a $\underline{U}$-category.
2. If $\underline{C}$ and $\underline{D}$ are categories with only one identity (i.e. semigroups), a functor $\underline{C} \to \underline{D}$ is just a semigroup homomorphism. When $\underline{C}$ and $\underline{D}$ furthermore are preadditive (i.e. rings), an additive functor is the same as a ring homomorphism.
3. If $\underline{C}$ and $\underline{D}$ are preordered sets, then a functor $\underline{C} \to \underline{D}$ is an order-preserving function.
4. The "forgetful" functor (Ab) $\to$ (Ens) which takes an abelian group to its underlying set.
5. There exists an imbedding F: (Ens) $\to$ (Ab), where $F(A)$ = the free abelian group on the set A, and $F(\alpha)$ = the induced homomorphism obtained by linear extension from the basis.
6. For every $\underline{U}$-small category $\underline{C}$ there exists an imbedding
   $T: \underline{C} \to \underline{U}$ - (Ens) with
   $T(A) = \{\xi: X \to A\}$
   $T(\alpha): \xi \mapsto \alpha\xi$ for $\alpha: A \to B$.
7. If $\underline{C}$ is a subcategory of $\underline{D}$, then the inclusion map $\underline{C} \to \underline{D}$ is a functor which is an imbedding. This functor is full if and only if $\underline{C}$ is a full subcategory of $\underline{D}$.

   Conversely, if $T: \underline{C} \to \underline{D}$ is an imbedding, then $\underline{C}$ is isomorphic to a subcategory of $\underline{D}$.

8. If $T: \underline{C} \to \underline{D}$ is a full imbedding, we may consider the full subcategory $\underline{D}'$ of $\underline{D}$ consisting of those objects which are isomorphic to objects of the form $T(A)$. T induces an equivalence between $\underline{C}$ and $\underline{D}'$.
9. Let $\underline{C}$ be the category of affine algebraic varieties over a fixed ground field k. The functor which to each variety associates its coordinate ring defines an equivalence between $\underline{C}^o$ and the category of finitely generated commutative k-algebras without zero-divisors [19].
10. A functor $\underline{C}^o \to \underline{D}$ is often called a contravariant functor from $\underline{C}$ to $\underline{D}$. Ordinary functors $\underline{C} \to \underline{D}$ are traditionally called covariant functors.

When dealing with "functors of several variables" it is most convenient to use product categories. Let $(C_i)_I$ be a set of categories. The product category $\prod_I \underline{C}_i$ is defined as the set theoretical product of the sets $\underline{C}_i$, with binary composition defined component-wise; the category axioms C 1 - 3 are easily verified. We then have:

$$\mathrm{Ob}\,(\Pi\, \underline{C}_i) = \Pi\, \mathrm{Ob}\,(\underline{C}_i)$$

$$\mathrm{Hom}_{\Pi \underline{C}_i}\,((A_i)_I,\ (B_i)_I) = \Pi\, \mathrm{Hom}_{\underline{C}_i}\,(A_i,\ B_i).$$

If each of the categories $\underline{C}_i$ is preadditive, then clearly also $\Pi\, \underline{C}_i$ is preadditive. If every $\underline{C}_i$ is abelian, then it is easily seen that also $\Pi\, \underline{C}_i$ is abelian. (In general, kernels, products, images etc. in the product category are obtained by taking them in each component category).

Let us in particular consider the case of a product category $\underline{C}_1 \times \underline{C}_2$. Let $T: \underline{C}_1 \times \underline{C}_2 \to \underline{D}$ be a functor. For fixed objects A in $\underline{C}_1$ and B in $\underline{C}_2$ we obtain "partial" functors:

$T_1: \underline{C}_1 \to \underline{D}$ defined as $T_1\,(X) = T\,(X,B)$

$T_1\,(\alpha) = T\,(\alpha,\ 1_B)$,

$T_2: \underline{C}_2 \to \underline{D}$ defined as $T_2\,(Y) = T(A,Y)$

$T_2\,(\beta) = T(1_A,\ \beta)$.

T is a functor $\Leftrightarrow T_1$ and T_2 are functors (for any choice of A and B) and for any morphisms $\alpha: A \to A'$ in $\underline{C}_1$ and $\beta: B \to B'$ in $\underline{C}_2$, the following diagram is commutative:

$$\begin{array}{ccc} T\,(A,B) & \longrightarrow & T\,(A,B') \\ \downarrow & & \downarrow \\ T\,(A',B) & \longrightarrow & T\,(A',B') \end{array}$$

The Hom functor:

Let $\underline{C}$ be a $\underline{U}$ -category. (It should be noted here that given any category $\underline{C}$ there exists a universe $\underline{U}$ such that $\underline{C}$ is a $\underline{U}$-category). We want to show that Hom may be considered as a functor $\underline{C}^{o} \times \underline{C} \to \underline{U}$ - (Ens).

For a fixed object A we define a functor $h^A: \underline{C} \to \underline{U}$ - (Ens) as:
$h^A(B) = \text{Hom}\,(A,B)$,
for $\beta: B \to B'$, $h^A\,(\beta)$: Hom $(A,B) \to$ Hom (A,B') is $\xi \mapsto \beta\xi$.
h^A is clearly a functor. Similarly for a fixed B there is a functor
$h^B: \underline{C}^{o} \to \underline{U}$ - (Ens):
$h_B\,(A) = \text{Hom}\,(A,B)$,
for $\alpha: A' \to A$, $h_B(\alpha)$: Hom $(A,B) \to$ Hom(A',B) is $\xi \mapsto \xi\alpha$.

Given both $\alpha: A' \to A$ and $\beta: B \to B'$ we obtain a diagram

$$\begin{array}{ccc} \text{Hom}\,(A,B) & \longrightarrow & \text{Hom}(A,B') \\ \downarrow & & \downarrow \\ \text{Hom}\,(A',B) & \longrightarrow & \text{Hom}\,(A',B') \end{array}$$

which is commutative since both ways give $\xi \mapsto \beta\xi\alpha$. It follows that h^A and h_B are partial functors of a functor
Hom: $\underline{C}^{o} \times \underline{C} \to \underline{U}$ - (Ens).

When $\underline{C}$ is preadditive, Hom is an additive functor $\underline{C}^{o} \times \underline{C} \to \underline{U}$ - (Ab).

Exercises:

Let $T: \underline{C} \to \underline{D}$ be a functor. Show that:

1. T preserves retractions and coretractions.

2. If T is faithful and α is a morphism in $\underline{C}$ such that $T(\alpha)$ is a monomorphism, then α is a monomorphism.
3. Show that if $(\underline{C}_i)_I$ is a family of $\underline{U}$ - small categories and $I \in \underline{U}$, then $\prod_I \underline{C}_i$ is a product for the family in $\underline{U}$ - (Cat).

§ 2. Exactness properties of additive functors.

Throughout this § we assume that $\underline{C}$ and $\underline{D}$ are abelian categories (generalizations to arbitrary categories will be studied in ch. 3).

Lemma. Every additive functor $T: \underline{C} \to \underline{D}$ preserves zero objects and zero morphisms.

Proof. Consider any zero morphism $o: A \to B$ in $\underline{C}$. Then $T(o) = T(o + o) = T(o) + T(o)$ implies $T(o) = o$, so T preserves zero morphisms. Since a zero object is characterized by the property that its identity morphism is zero, it follows that T preserves zero objects.

Proposition 1. A functor $T: \underline{C} \to \underline{D}$ is additive if and only if it preserves finite products.

Proof. Suppose T is additive and consider a product diagram

$$A \underset{u_1}{\overset{p_1}{\rightleftarrows}} A \times B \underset{u_2}{\overset{p_2}{\rightleftarrows}} B$$

where $p_j u_i = \delta_{ij}$ and $u_1 p_1 + u_2 p_2 = 1$. Then $T(p_j)T(u_i) = T(p_j u_i) = \delta_{ij}$ and $T(u_1)T(p_1) + T(u_2)T(p_2) = T(u_1 p_1 + u_2 p_2) = T(1) = 1$. It follows from ch. 1, Prop. 5 that T carries the given diagram into a product diagram.

The converse statement follows from the description of addition of morphisms given at the end of ch. 1, § 6.

Note that the proof also gives:

Corollary. An additive functor preserves split exactness of short exact sequences.

In the following, let $T: \underline{C} \to \underline{D}$ be an additive functor.

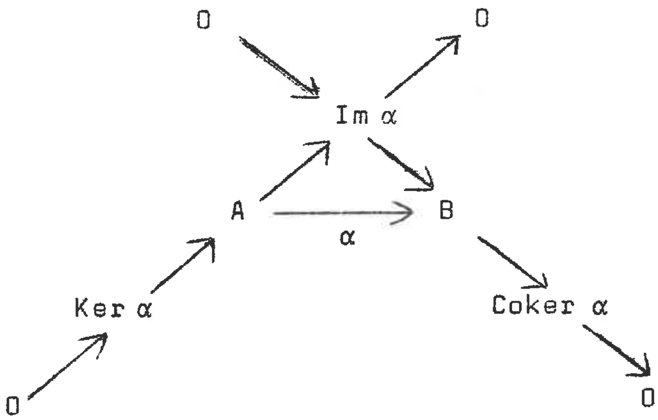
Proposition 2. The following properties of T are equivalent:

a) If $0 \to A \to B \to C \to 0$ is a s.e.s. in $\underline{C}$, then $0 \to T(A) \to T(B) \to T(C)$ is exact in $\underline{D}$.

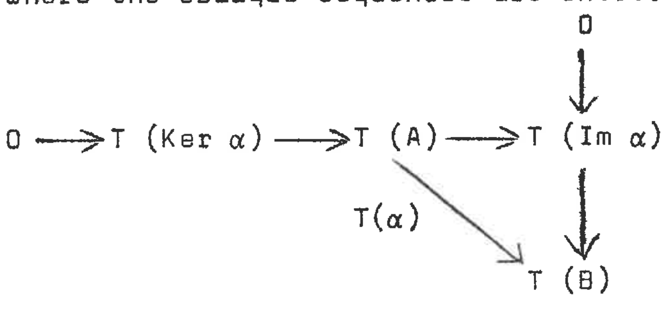
b) T preserves kernels, i.e. $T(\ker \alpha) = \ker T(\alpha)$ for any morphism α in $\underline{C}$.

Definition. A functor T with these properties is left exact.

Proof. Note that a sequence $0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C$ is exact if and only if $\alpha = \ker \beta$. Hence it is clear that (a) is a special case of (b).
a) $\Rightarrow$ b): Let $\alpha: A \to B$ in $\underline{C}$. There is a commutative diagram



where the oblique sequences are exact. From this we obtain



with exact row and column. This gives $T(\ker \alpha) = \ker T(\alpha)$ (using exercise 4 of ch. 1, § 7).

Corollary. A left exact functor preserves monomorphisms.
There is of course a dual notion of right exact functor. Prop. 2 and its dual together give:

Proposition 3. The following properties of T are equivalent:

a) T preserves kernels and cokernels.

b) T carries s.e.s. into s.e.s.

c) T carries arbitrary exact sequences into exact sequences.

Definition. A functor T with these properties is exact.

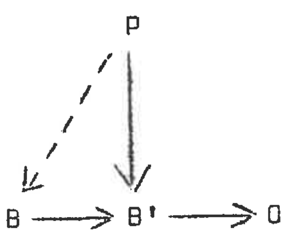
Proof. It only remains to prove a) $\Rightarrow$ c). If $A \xrightarrow{\alpha} B \xrightarrow{\alpha} C$ is exact in $\underline{C}$, then $\ker(\operatorname{coker}\alpha) = \ker\beta$. (a) then implies
$\ker(\operatorname{coker} T(\alpha)) = T(\ker(\operatorname{coker}\alpha)) = T(\ker\beta) = \ker T(\beta)$, so
$T(A) \to T(B) \to T(C)$ is exact.

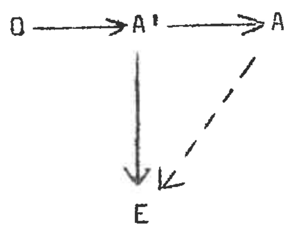
Examples:

1. Let $\underline{C}$ be a subcategory of $\underline{D}$ (both categories still being abelian). $\underline{C}$ is said to be an abelian subcategory of $\underline{D}$ if the inclusion functor is additive and exact. Clearly this is equivalent to saying that the preadditive structure of $\underline{C}$ is induced from that of $\underline{D}$, and that if α is a morphism of $\underline{C}$ then its kernel and cokernel in $\underline{D}$ are actually objects in $\underline{C}$.
2. The functors $h^A: \underline{C} \to (Ab)$ and $h_B: \underline{C}^o \to (Ab)$ are both left exact, as follows directly from the definition of kernel and cokernel.

Definition. An object P of $\underline{C}$ is called projective if h^P is an exact functor. Dually, an object E is called injective if h_E is exact.

P is projective if and only if for every epimorphism $\beta: B \to B'$ and morphism $\varphi': P \to B'$ there exists $\varphi: P \to B$ such that $\beta\varphi = \varphi'$; dually for injectives.





Proposition 4. An object P is projective if and only if every s.e.s. $0 \to A \to B \to P \to 0$ splits.

Proof. If P is projective, then 1_P: $P \to P$ may be lifted to a morphism $P \to B$ such that $1_P = P \to B \to P$. Hence $B \to P$ is a retraction.

Conversely, suppose every epimorphism $B \to P$ is a retraction. If β: $B \twoheadrightarrow B'$ is an epimorphism and φ: $P \to B'$ is given, we look at the pullback diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & P \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ B & \xrightarrow[\beta]{} & B' \end{array}$$

where also α is an epimorphism by the Prop. of Appendix 1.A. Hence there exists ξ: $P \to A$ such that $\alpha\xi = 1$. Then $\psi\xi$: $P \to B$ has the property that $\beta\psi\xi = \varphi\alpha\xi = \varphi$, as desired.

Proposition 5. i) A coproduct $\bigoplus A_i$ is a projective object if and only if each A_i is projective.
ii) A product $\prod_I B_i$ is an injective object if and only if each B_i is injective.
Proof. Easy.

The category is said to "have enough projectives" if every object is a quotient object of a projective object; dually, it "has enough injectives" if every object is a subobject of an injective object. The existence of enough prouectives or enough injectives is of course of value when one wants to carry on homological algebra in the category.

Let us finally consider faithful functors in the additive case. Since T: $\underline{C} \to \underline{D}$ has been assumed additive, it is clear that it is faithful if and only if $T(\alpha) \neq o$ for every non-zero morphism α in $\underline{C}$.

Proposition 6. Suppose T is exact. T is then faithful if and only if $T(A) = 0 \Rightarrow A = 0$.

Proof. If T is faithful and $T(A) = 0$, then $T(1_A) = o = T(o_{A,A})$, so $1_A = o$ and $A = 0$. Converse: if $\alpha \neq o$, then $\operatorname{Im}\alpha \neq 0$, so $T(\operatorname{Im}\alpha) \neq 0$ by hypothesis. But $T(\operatorname{Im}\alpha) = \operatorname{Im} T(\alpha)$ by exactness, hence $T(\alpha) \neq o$.

Definition. An object A in $\underline{C}$ is called a generator for $\underline{C}$ if h^A is faithful, and a cogenerator for $\underline{C}$ if h_A is faithful.
A is a generator $\Leftrightarrow$ for every non-zero $\alpha: B \to C$, also $\mathrm{Hom}(A,B) \to \mathrm{Hom}(A,C)$ is $\neq 0$, i.e. there exists $\varphi: A \to B$ such that $\alpha\varphi \neq 0$. Prop. 6 gives:

Proposition 7. A projective object P is a generator if and only if there exists a non-zero morphism $P \to A$ for each $A \neq 0$.

Proposition 8. Suppose $\underline{C}$ is a $\underline{U}$-category and has $\underline{U}$-coproducts. An object G is a generator if and only if for each object A there is an epimorphism $G^I \to A$ for some $I \in \underline{U}$.

Proof. Put $I = \mathrm{Hom}(G,A)$ and let $\varphi: G^I \to A$ be defined by the family $(\varphi_\gamma)_{\gamma \in I}$ where $\varphi_\gamma = \gamma: G \to A$. (The notation G^I was introduced in ch. 1, § 4). φ is an epimorphism $\Leftrightarrow$ $\alpha\varphi \neq 0$ for every non-zero $\alpha: A \to B$ $\Leftrightarrow$ for every non-zero $\alpha: A \to B$ there exists $\gamma: G \to A$ such that $\alpha\gamma \neq 0$. From this the conclusion is immediate.

Corollary. If $\underline{C}$ is a $\underline{U}$-category with $\underline{U}$-coproducts and a projective generator, then $\underline{C}$ has enough projectives.

Exercise:
Show that if an abelian category has enough projectives and a generator, then it has a projective generator.

§ 3. Projective and injective modules.

In this § we will study in somewhat more detail the projective and the injective objects in the category of right modules over a ring A.

Proposition 9. A right A-module P is projective if and only if P is a direct summand of a free module.

Proof. $\Rightarrow$: Write P as a quotient of a free module and apply Prop. 4.
$\Leftarrow$: Suppose $P \oplus K = F$ with F a free module. If $\beta: L \to M$ is an epimorphism and $\varphi: P \to M$ is given, we may extend φ to $\varphi': F \to M$,

e.g. by putting $\varphi'(K) = 0$. Let $(x_i)_I$ be a basis for F, and choose $y_i \in L$ such that $\beta(y_i) = \varphi'(x_i)$. A well-defined morphism $\psi: P \to L$ is then obtained by letting $x_i \mapsto y_i$, and $\beta\psi = \varphi$.

Corollary. There are enough projective objects in $\mathrm{Mod}\,(A)$. •

The preceding description of projective modules should be compared with the following characterization of generators:

Proposition 10. A right A-module P is a generator if and only if A is a direct summand of a direct sum of copies of P.
Proof. Quite easily seen to be a consequence of Prop. 8.

Examples.

1. If A is a principal ideal domain, then every submodule of a free module is free ([8], p. 134) and hence every projective module is free. In particular this applies to the category (Ab).
2. Every finitely generated free module is a projective generator.
3. The ring $\underline{Z}/6\underline{Z}$ may be decomposed as $\underline{Z}/6\underline{Z} = (\bar{3}) \oplus (\bar{2})$. Each of these two ideals is a cyclic projective module which is not free.

It is a little more complicated to establish the existence of enough injective modules. Let us first consider the case of $\underline{Z}$-modules ($\underline{Z}$ stands for the integers, $\underline{Q}$ for the rationals).

Proposition 11. An abelian group G is injective if and only if it is divisible (i.e. $G = nG$ for every integer $n \neq 0$).
Proof. Suppose G is injective, and $x \in G$ is arbitrary. The diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & n\underline{Z} & \longrightarrow & \underline{Z} \\ & & \varphi \big\downarrow & & \\ & & G & & \end{array} \qquad \varphi(nm) = mx$$

may be completed with $\varphi': \underline{Z} \to G$ so that $\varphi'(n) = x$. Then $x = n\,\varphi'(1)$ and so x is divisible by n.

To prove the converse statement, we assume G to be divisible and consider

$$\begin{array}{ccccc} 0 & \longrightarrow & L & \longrightarrow & M \\ & \varphi & \big\downarrow & & \\ & & G & & \end{array}$$

Using a conventional Zorn's lemma type of argument, we are reduced to the case when φ may not be extendend further to a subgroup of M, and then want to prove $L = M$. If $x \in M$ but $x \notin L$, then we must clearly have $L \cap \underline{Z}x \neq (0)$. Let n be the smallest integer > 0 such that $nx \in L$. Choose $g \in G$ such that $\varphi(nx) = ng$. If we put $\varphi'(y + mx) = \varphi(y) + mg$ for $y \in L$, then it is easily verified that φ' is a well-defined proper extension of φ. This is a contradiction, so $L = M$.

<u>Lemma</u> $\underline{Q}/\underline{Z}$ is an injective cogenerator for (Ab).

<u>Proof</u>. $\underline{Q}/\underline{Z}$ is clearly divisible, so it is injective by the proposition. By Prop. 6* it remains to show that there exists a non-zero morphism $G \to \underline{Q}/\underline{Z}$ for each abelian group $G \neq (0)$, and since $\underline{Q}/\underline{Z}$ is injective it even suffices to do this for cyclic G. But every cyclic group may be imbedded in $\underline{Q}/\underline{Z}$.

<u>Proposition 12</u>. Mod (A). has an injective cogenerator, and therefore has enough injectives.

<u>Proof</u>. For any right A-module M and abelian group G we have natural isomorphisms $\mathrm{Hom}_A(M, \mathrm{Hom}_{\underline{Z}}(A,G)) \cong \mathrm{Hom}_{\underline{Z}}(M \otimes_A A, G) \cong \mathrm{Hom}_{\underline{Z}}(M,G)$, where A and M are considered as $\underline{Z}$ - A - bimodules and $\mathrm{Hom}_{\underline{Z}}(A,G)$ is a right A-module by letting $\varphi a\colon a' \mapsto \varphi(aa')$. (Cf. [17], Ch. V. 3 for details). If we in particular choose $G = \underline{Q}/\underline{Z}$, then $\mathrm{Hom}_{\underline{Z}}(\cdot\,, G)$ is an exact and faithful functor, which composed with the forgetful functor Mod(A). $\to$ (Ab), which also is exact and faithful, gives the functor $\mathrm{Hom}_A(\cdot\,, \mathrm{Hom}_{\underline{Z}}(A,G))$. It follows that the later functor also is exact and faithful, i.e. $\mathrm{Hom}_{\underline{Z}}(A, \underline{Q}/\underline{Z})$ is an injective cogenerator for Mod (A). .

Proposition 13. An injective right A-module is a cogenerator if and only if it contains an isomorphic copy of each simple right A-module.

Proof. If E is an injective cogenerator and S is simple, then there exists a non-zero morphism $S \to E$ which trivially must be a monomorphism.

To prove the converse assertion, we note that every cyclic module has a simple quotient module, since every ideal is contained in a maximal ideal. We then argue as in the preceding lemma.

Exercises:

1. Let A be an integral domain. Show that:
   i) every injective module E is divisible (i.e. $E = aE$ for each $a \neq o$ in A).
   ii) The class of divisible modules is closed under direct sums and quotients.
2. Let $\underline{C}$ be the category of abelian torsion groups. Show that:
   i) $\underline{C}$ is an abelian subcategory of (Ab).
   ii) $\underline{C}$ has a generator and an injective cogenerator, but no projective objects $\neq 0$.
3. Let A be a right noetherian ring and let $\underline{C}$ be the full subcategory of Mod (A). consisting of the finitely generated modules. Show that:
   i) $\underline{C}$ is an abelian subcategory of Mod (A). .
   ii) $\underline{C}$ has enough projectives.

§ 4. Functor categories.

Let $\underline{C}$ and $\underline{D}$ be two arbitrary categories. We will define a new category, whose objects are the functors from $\underline{C}$ to $\underline{D}$, and whose morphisms are the "natural transformations" of functors.

Definition. Let $S, T: \underline{C} \to \underline{D}$ be two functors. A natural transformation $\eta: S \to T$ is obtained by taking for each object A in $\underline{C}$ a morphism $\eta_A: S(A) \to T(A)$ in $\underline{D}$, so that for every morphism $\alpha: A \to B$ in $\underline{C}$, the following diagram commutes:

$$\begin{array}{ccc} S(A) & \xrightarrow{\eta_A} & T(A) \\ {\scriptstyle S(\alpha)}\downarrow & & \downarrow{\scriptstyle T(\alpha)} \\ S(B) & \xrightarrow[\eta_B]{} & T(B) \end{array}$$

Two natural transformations $\xi: R \to S$, $\eta: S \to T$ between functors $R, S, T: \underline{C} \to \underline{D}$ may be composed in an evident way to give a natural transformation $\eta\xi: R \to T$. This composition is associative. For every functor $T: \underline{C} \to \underline{D}$ there is an identity transformation $1_T: T \to T$. It is clear that in this way we obtain a category Fun $(\underline{C}, \underline{D})$, whose objects are functors $\underline{C} \to \underline{D}$ and where Hom (S, T) is the set of natural transformations $S \to T$ (we will usually write Nat (S, T) instead of Hom (S, T)).

If $\underline{C}$ is $\underline{U}$-small and $\underline{D}$ is a $\underline{U}$-category, then Fun $(\underline{C}, \underline{D})$ is a $\underline{U}$-category, because Nat $(S, T) \subset \prod_{\mathrm{Ob}\,\underline{C}} \mathrm{Hom}_{\underline{D}}\,(S(A), T(A))$.

The isomorphisms in Fun $(\underline{C}, \underline{D})$ are called natural equivalences; clearly $\eta: S \to T$ is a natural equivalence if and only if η_A is an isomorphism in $\underline{D}$ for each A.

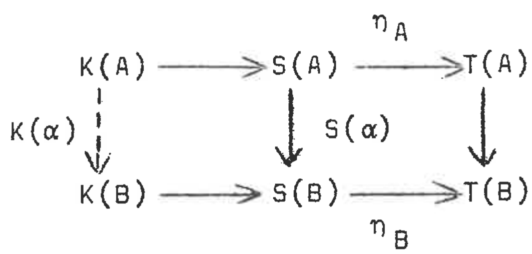
It may be said as a general rule that the category Fun$(\underline{C}, \underline{D})$ inherits the good properties of $\underline{D}$. We give some examples of this:

1) If $\underline{D}$ has zero objects, then so has obviously Fun $(\underline{C}, \underline{D})$.

2) If $\underline{D}$ is preadditive, then so is Fun $(\underline{C}, \underline{D})$.
   For let $\xi, \eta: S \to T$ be natural transformations. Then $\xi + \eta$ is given as $(\xi + \eta)_A = \xi_A + \eta_A$. One verifies that this makes Nat (S, T) an abelian group.

3) If $\underline{D}$ has finite products, then so has Fun $(\underline{C}, \underline{D})$.
   Let $T_1, \ldots, T_n$ be functors $\underline{C} \to \underline{D}$. Define a new functor T as
   $T(A) = T_1(A) \times \ldots \times T_n(A)$,
   $T(\alpha) = T_1(\alpha) \times \ldots \times T_n(\alpha)$ for $\alpha: A \to B$.
   One verifies that $T = T_1 \times \ldots \times T_n$.

4) If $\underline{D}$ has kernels or cokernels, then so has Fun $(\underline{C}, \underline{D})$.
   Let $\eta: S \to T$ be a natural transformation. Define a functor $K: \underline{C} \to \underline{D}$ as
   $K(A) = \mathrm{Ker}\,\eta_A$.
   $K(\alpha)$ = induced by $S(\alpha)$, for $\alpha: A \to B$.



One verifies that $K \to S$ is a kernel for η.

Proposition 14. If $\underline{D}$ is an abelian category, then also Fun $(\underline{C}, \underline{D})$ is abelian.

Proof. AB 1 and 2 are clear from the remarks above. AB 3 is also quite clear since it holds "pointwise".

Diagram categories:

As a simple example, consider diagrams of the general form $A \to B \to C$ in $\underline{D}$. A "morphism" between two such diagrams $A \to B \to C$ and $A' \to B' \to C'$ is defined to be a triple of morphisms $A \to A'$, $B \to B'$, $C \to C'$ such that the diagram

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

commutes. The set of diagrams of this form then makes up a category.

A better way of handling such "diagram categories" is to consider them as particular cases of functor categories. In the actual example we let $\underline{I}$ be the category consisting of just three objects a, b, c and morphisms α: $a \to b$, β: $b \to c$, $\beta\alpha$: $a \to c$ plus the identities. A diagram $A \to B \to C$ may be viewed as a functor $\underline{I} \to \underline{D}$. A morphism between diagrams is just a natural transformation of such functors. Hence the category of diagrams is isomorphic to the functor category Fun $(\underline{I}, \underline{D})$.

In a similar way one can handle more complicated diagrams, including diagrams with commutativity relations (but not with exactness conditions!). For example, a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is a functor to $\underline{D}$ from the category

$$\begin{array}{ccc} a & \xrightarrow{\alpha} & b \\ {\scriptstyle\gamma}\downarrow & \searrow^{\varepsilon} & \downarrow{\scriptstyle\beta} \\ c & \xrightarrow[\delta]{} & d \end{array} \qquad \begin{array}{l} \beta\alpha = \delta\gamma = \varepsilon \\ + \text{ identities.} \end{array}$$

On the other hand, every functor category Fun ($\underline{C}$, $\underline{D}$) may be considered (though somewhat artificially) as a diagram category. For this reason we will not develop any special theory of diagram categories (the reader is referred to [18], p. 42, for a treatment of diagram categories and commutativity relations).

Additive functors:

Suppose both $\underline{C}$ and $\underline{D}$ are preadditive categories. The full subcategory of Fun ($\underline{C}$, $\underline{D}$) consisting of the additive functors is denoted by Hom ($\underline{C}$, $\underline{D}$). This category also inherits properties from $\underline{D}$. E.g., if $\underline{D}$ has kernels then if $\eta: S \to T$ is a morphism in Hom ($\underline{C}$, $\underline{D}$), the functor K that was constructed in 4) above will be additive and a kernel for η also in Hom ($\underline{C}$, $\underline{D}$). We clearly have:

Proposition 15 If $\underline{C}$ is preadditive and $\underline{D}$ is abelian, then Hom ($\underline{C}$, $\underline{D}$) is an abelian subcategory of Fun ($\underline{C}$, $\underline{D}$).

Module categories:

Let A be a ring, i.e. a preadditive category with only one identity (which we denote by 1). If $\underline{D}$ is an arbitrary preadditive category, an object T in Hom (A, $\underline{D}$) is called a left A-object in $\underline{D}$. T determines an object B in $\underline{D}$ by the rule $T(1) = 1_B$, and for each $\alpha \in A$, a morphism $T(\alpha): B \to B$ with the properties:

$$T(\beta\alpha) = T(\beta)\, T(\alpha)$$

$$T(\alpha + \beta) = T(\alpha) + T(\beta).$$

Hence T induces a ring homomorphism $A \to \mathrm{Hom}_{\underline{D}}(B, B)$. Conversely, every pair (B, μ), where B is an object of $\underline{D}$ and μ a homomorphism $\mu: A \to \mathrm{Hom}_{\underline{D}}(B, B)$ of rings with identities, determines uniquely a left A-object in $\underline{D}$.

Dually, we call Hom (A^o, $\underline{D}$) the category of right A-objects in $\underline{D}$.

Let us now consider the particular case of $\underline{D}$ = (Ab). Let (B, μ) be a left A-object in (Ab). Write $\mu(\alpha)\, x = \alpha x$. Then we have

$$1x = x$$
$$(\beta\alpha)x = \beta(\alpha x)$$
$$(\alpha + \beta)x = \alpha x + \beta x$$
$$\alpha(x + y) = \alpha x + \alpha y,$$

so μ endows B with a structure of left A-module. It is clear that we can identify the notions of left A-object in (Ab) and left A-module.

It remains to consider the morphisms in $\underline{\text{Hom}}\,(A, \underline{D})$ for arbitrary $\underline{D}$. Let $T = (B, \mu)$ and $S = (C, \mu')$ be left A-objects in $\underline{D}$. A natural transformation $\eta: T \to S$ corresponds to a morphism $B \to C$ such that for each $\alpha \in A$, the diagram

$$\begin{array}{ccc} B & \longrightarrow & C \\ {\scriptstyle \mu(\alpha)}\downarrow & & \downarrow{\scriptstyle \mu'(\alpha)} \\ B & \longrightarrow & C \end{array}$$

commutes. In case $\underline{D} = (Ab)$ this means that $B \to C$ is an A-linear map. Hence we may identify

$$\text{Mod}_{\bullet}\,(A) = \underline{\text{Hom}}\,(A, (Ab))$$
$$\text{Mod}\,(A)_{\bullet} = \underline{\text{Hom}}\,(A^{o}, (Ab)).$$

Exercises:

Let $\underline{B}$, $\underline{C}$ and $\underline{D}$ be categories.

1. Let $\eta: S \to T$ be a morphism in $\text{Fun}\,(\underline{C}, \underline{D})$. Show that if η_A is a monomorphism for each object A in $\underline{C}$, then η is a monomorphism. Show that the converse holds when $\underline{D}$ is abelian.
2. Let $\underline{D}$ be abelian. Show that a sequence $0 \to R \to S \to T \to 0$ in $\text{Fun}\,(\underline{C}, \underline{D})$ is exact if and only if $0 \to R(A) \to S(A) \to T(A) \to 0$ is exact for each object A in $\underline{C}$.
3. Define a functor $E: \text{Fun}\,(\underline{C}, \underline{D}) \times \underline{C} \to \underline{D}$ as $E(T, A) = T(A)$, etc. The partial functor E_A for fixed A is the evaluation functor at A. Show that:
   i) if $\underline{D}$ is preadditive, then E_A is additive;
   ii) if $\underline{D}$ is abelian, then E_A is exact.
4. Show that there are canonical isomorphisms of categories
   $\text{Fun}\,(\underline{B}, \text{Fun}\,(\underline{C},\underline{D})) \cong \text{Fun}\,(\underline{B} \times \underline{C}, \underline{D}) \cong \text{Fun}\,(\underline{C}, \text{Fun}\,(\underline{B}, \underline{D}))$.
   If $\underline{B}$, $\underline{C}$ and $\underline{D}$ are preadditive, then a similar result holds for $\underline{\text{Hom}}$.

§ 5. Representable functors.

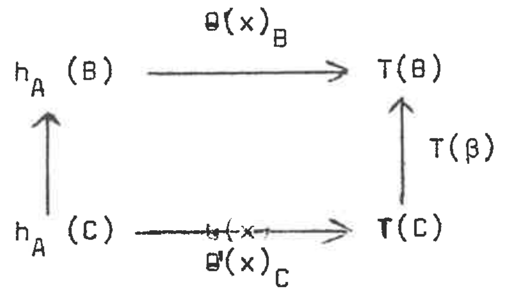
As we saw in the preceding §, the category Fun ($\underline{C}$, (Ens)) has many of the good properties of the category (Ens). It is therefore an interesting fact that there is a full imbedding of $\underline{C}$ into Fun (C^{o},(Ens)). This is a corollary of the following more general theorem:

Theorem 16 (Yoneda). Let $\underline{C}$ be a $\underline{U}$-category. For every object A in $\underline{C}$ and every functor $T: \underline{C}^{o} \to \underline{U}$-(Ens) there is a bijection

$$\theta_{A,T}: \text{Nat}\ (h_A, T) \to T(A)$$

which is natural in A and T.
If $\underline{C}$ is preadditive and $T: \underline{C}^{o} \to \underline{U}$ - (Ab) is an additive functor, then $\theta_{A,T}$ is a group isomorphism.

Proof. To define θ we note that a natural transformation $\eta: h_A \to T$ in particular determines a map $\eta_A: h_A(A) \to T(A)$, and we put $\theta_{A,T}\ (\eta) = \eta_A\ (1_A)$. An inverse θ' of θ is constructed as follows: if $x \in T(A)$, then $\theta'(x) \to T$ is defined as
$\theta'(x)_B: \alpha \mapsto T\ (\alpha)\ (x)$ for $\alpha: B \to A$.
The rest of the proof consists of a number of mechanical verifications.
1) $\theta'(x)$ is a natural transformation:
For every $\beta: B \to C$ we have a diagram



which should be commutative. Starting with $\alpha \in h_A(C)$ and passing over the NW corner gives $\theta'(x)_B\ (\alpha\beta) = T(\alpha\beta)\ (x)$, while the SE route gives $T(\beta)\ \theta'(x)_C(x) = T(\beta)\ T(\alpha)(x) = T\ (\alpha\beta)\ (x)$. Thus the diagram commutes.
2) $\theta'\ \theta = \text{id.}$:
For arbitrary $\eta: h_A \to T$ and $\alpha: B \to A$ we have
$(\theta'\ \theta\ (\eta))_B\ (\alpha) = (\theta'\ \eta_A\ (1_A))_B\ (\alpha) = T(\alpha)\ \eta_A\ (1_A) = \eta_B\ (\alpha)$,
where the last equality follows from the commutativity of the diagram

$$\begin{array}{ccc} h_A(A) & \xrightarrow{\eta_A} & T(A) \\ \downarrow & & \downarrow T(\alpha) \\ h_A(B) & \xrightarrow[\eta_B]{} & T(B) \end{array}$$

3) $\theta\,\theta' = \mathrm{id.}$:

For any $x \in T(A)$ we get $\theta\,\theta'(x) = \theta'(x)_A\,(1_A) = T\,(1_A)\,(x) = x.$

4) Naturality in A:

Given any $\alpha\colon B \to A$, we have to show that the diagram

$$\begin{array}{ccc} \mathrm{Nat}\,(h_A, T) & \xrightarrow{\theta_{A,T}} & T(A) \\ \downarrow & & \downarrow \\ \mathrm{Nat}\,(h_B, T) & \xrightarrow[\theta_{B,T}]{} & T(B) \end{array}$$

commutes, where the left map is induced by $h_\alpha = \mathrm{Hom}\,(\cdot, \alpha)\colon h_B \to h_A$. Start with $\eta\colon h_A \to T$. SW route gives $\theta_{B,T}\,(\eta \cdot h_\alpha) = (\eta \cdot h_\alpha)_B\,(1_B) =$

$= \eta_B (h_\alpha)_B\,(1_B) = \eta_B\,\mathrm{Hom}\,(1_B, \alpha)\,(1_B) = \eta_B\,(\alpha)$, while the NE route gives $T(\alpha)\,\theta_{A,T}\,(\eta) = T(\alpha)\,\eta_A\,(1_A) = \eta_B(\alpha)$ as in (2).

5) Naturality in T:

Given any natural transformation $\xi\colon S \to T$, we have to show that the diagram

$$\begin{array}{ccc} \mathrm{Nat}\,(h_A, S) & \xrightarrow{\theta_{A,S}} & S(A) \\ \downarrow & & \downarrow \xi_A \\ \mathrm{Nat}\,(h_A, T) & \xrightarrow[\theta_{A,T}]{} & T(A) \end{array}$$

commutes. Starting with $\eta\colon h_A \to S$, the NE route takes η to $\xi_A(\theta_{A,S}(\eta)) = \xi_A(\eta_A\,(1_A)) = (\xi\eta)_A\,(1_A)$, while the SW route takes η to $\theta_{A,T}\,(\xi\eta) = (\xi\eta)_A\,(1_A)$.

6) In the additive case, $\theta_{A,T}$ is clearly a group homomorphism. This concludes the proof of the theorem.

Let us now consider an important special case of the Yoneda theorem, namely that in which $T = h_B$ for some object B. The theorem asserts that there is an isomorphism θ: Nat $(h_A, h_B) \to$ Hom (A,B). Its inverse θ^{-1} was constructed in the proof as taking $\alpha: A \to B$ to the natural transformation $\theta^{-1}(\alpha)$ with $\theta^{-1}(\alpha)_C$: Hom $(C,A) \to$ Hom (C,B) given as $\gamma \mapsto \alpha\gamma$. We will write h_α for $\theta^{-1}(\alpha)$, and obtain

Corollary. For every $\underline{U}$-category $\underline{C}$ there is a full imbedding $\underline{C} \to$ Fun $(\underline{C}^o, \underline{U}$-(Ens)) given by $A \mapsto h_A$, $\alpha \mapsto h_\alpha$. When $\underline{C}$ is preadditive, there is a full imbedding $\underline{C} \to \underline{\text{Hom}}\ (\underline{C}^o, \underline{U}$ - (Ab)).

Definition. A functor $T: \underline{C}^o \to$ (Ens) is representable if there exists a natural equivalence $\eta: h_A \to T$ for some A.

The functor $A \mapsto h_A$ defines an equivalence between $\underline{C}$ and the full subcategory of Fun $(\underline{C}^o$, (Ens)) consisting of the representable functors.

By the Yoneda theorem, a natural equivlaence $\eta: h_A \to T$ corresponds to an element $x = \eta_A(1_A) \in T(A)$, and we say that the pair (A,x) represents T. The representing pair is essentially unique, i.e. if both (A,x) and (B,y) represent T, then there exists an isomorphism $\alpha: A \to B$ such that $T(\alpha): x \mapsto y$. Note that η may be reconstructed from (A,x) as $\eta_B(\alpha) = T(\alpha)(x)$ for $\alpha: B \to A$.

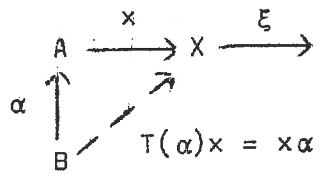
$$\begin{array}{ccc} \text{Hom}(A,A) & \xrightarrow{\eta_A} & T(A) \\ \downarrow & & \downarrow T(\alpha) \\ \text{Hom}(B,A) & \xrightarrow[\eta_B]{} & T(B) \end{array}$$

The natural transformation η is a natural equivalence if and only if η_B is bijective for each B, so we have the following characterization of representing pairs:

Let $T: \underline{C}^o \to$ (Ens). T is then represented by (A,x), where $x \in T(A)$, if and only if for every B the map η_B: Hom $(B,A) \to T(B)$, given as $\alpha \mapsto T(\alpha)(x)$, is bijective.

Example:

Let $\xi: X \to Y$ be a morphism in $\underline{C}$. Define $T: \underline{C}^o \to (Ens)$ as
$T(C) = \{\varphi: C \to X \mid \xi\varphi = o\}$,
$T(\gamma): T(C') \to T(C)$ as $\varphi \mapsto \varphi\gamma$ for $\gamma: C \to C'$.
It is easily seen that T is represented by a pair (A,x) if and only if $A = \text{Ker}\,\xi$ and $x = \ker \xi$.



$\xi\, x = o$

There is of course a dual theory for covariant functors $T: \underline{C} \to (Ens)$. The Yoneda theorem then asserts that there is a bijection $\text{Nat}\,(h^A, T) \to T(A)$. It is left for the reader to make explicit the dual conditions for representability of T.

Exercises:

Let $\underline{C}$ be a $\underline{U}$-category.

1. Redefine final objects in $\underline{C}$ by means of representability of a suitable functor $\underline{C}^o \to (Ens)$.
2. Let $\underline{U} \subset \underline{V}$ be universes. Show that:
   i) $\underline{U}$-(Ens) is a full subcategory of $\underline{V}$-(Ens).
   ii) There is a full imbedding
   $I: \text{Fun}\,(\underline{C}^o, \underline{U}-(Ens)) \to \text{Fun}\,(\underline{C}^o, \underline{V}-(Ens))$.
   iii) $F: \underline{C}^o \to \underline{U}$-(Ens) is representable if and only if $IF: \underline{C}^o \to \underline{V}$-(Ens) is representable (hence representability does not depend on the choice of universe).

Appendix II.A: Projectives and injectives in arbitrary categories.

In § 2 we introduced the notions of projective and injective objects for abelian categories. In the case of an arbitrary category $\underline{C}$ the corresponding definitions are:

Definition. An object P is projective if for each epimorphism $\alpha: A \to B$ and morphism $\varphi: P \to B$, there exists $\varphi': P \to A$ such that $\alpha\varphi' = \varphi$. Injective objects are defined dually.

Proposition 5 is valid also in this general case. It is also clear that if P is projective, then every epimorphism $A \to P$ is a retraction.

We may also define generators and cogenerators for $\underline{C}$. It will be useful, however, to introduce somewhat more general notions than was done in § 2.

Definition. A set of objects $\{G_i\}_I$ is a family of generators for $\underline{C}$ if for every pair of distinct morphisms $\alpha, \beta: A \to B$ there exists $\varphi: G_i \to A$ for some $i \in I$ such that $\alpha\varphi \neq \beta\varphi$. Cogenerators are defined dually.

If $\{G_i\}_I$ is a family of generators for $\underline{C}$ and if $\coprod_I G_i$ exists, then obviously $\coprod_I G_i$ will be a generator for $\underline{C}$. Prop. 8 holds also for arbitrary categories.

Examples:

1. Every object in (Ens) is projective, while every non-empty set is injective. Every non-empty set is also a generator and sets having more than one element are cogenerators.
2. The projective objects in (Gr) are the retracts of free groups. But since every subgroup of a free group is free, the projective groups are just the free groups. On the other hand, every injective group may be shown to be trivial, i.e. to be $\{1\}$ [7].

Appendix II.B: Group objects in categories.

It often happens that in a given category $\underline{C}$ one wants to consider objects with an additional algebraical structure. E.g. in (Top) one has the topological groups, topological rings etc. We will here study the group objects in a category $\underline{C}$. Let $F: (Gr) \to (Ens)$ be the forgetful functor.

Definition. A group object in $\underline{C}$ is a functor $\tilde{G}: \underline{C}^o \to (Gr)$ such that $F\tilde{G}$ is representable.

This means that there is a representing object $G \in \underline{C}$ such that for each $X \in \underline{C}$ one has a group structure on

$$\tilde{G}(X) \cong h_G(X) = \mathrm{Hom}(X,G),$$

and if $X \to Y$ is a morphism in $\underline{C}$, then the induced map $\mathrm{Hom}\,(Y,G) \to \mathrm{Hom}(X,G)$ is a group homomorphism.

An equivalent way of expressing this is to say that a group object in $\underline{C}$ is an object $G \in \underline{C}$ together with a natural transformation $\gamma: h_G \times h_G \to h_G$ satisfying the conditions of associativity etc.

Let us now assume that $\underline{C}$ has finite products. Then $h_G \times h_g$ is represented by $G \times G$, and $\gamma: h_G \times h_G \to h_G$ corresponds by the Yoneda theorem to a morphism $\mu: G \times G \to G$. The associativity condition then becomes:

1)
$$\begin{array}{ccc} G \times G \times G & \xrightarrow{1 \times \mu} & G \times G \\ {\scriptstyle \mu \times 1}\downarrow & & \downarrow{\scriptstyle \mu} \\ G \times G & \xrightarrow[\mu]{} & G \end{array}$$
commutes.

The condition that translations to the right and to the left should be invertible takes the form:

2) The two morphisms $G \times G \overset{\alpha}{\underset{\beta}{\rightrightarrows}} G \times G$ defined by

$$\begin{cases} \pi_1\alpha = \mu \\ \pi_2\alpha = \pi_2 \end{cases} \qquad\qquad \begin{cases} \pi_1\beta = \pi_1 \\ \pi_2\beta = \mu \end{cases}$$

are isomorphisms, where $\pi_1, \pi_2: G \times G \to G$ are the two canonical projections.

If $\underline{C}$ has a final object, then (2) may be reformulated in various ways, see e.g. [5] or [18].

The group objects of $\underline{C}$ form in a natural way a category $\underline{C}_{Gr}$.

Examples:

1. $(\mathrm{Ens})_{Gr}$ is just the category of groups.

2. If $\underline{C}$ is an additive category, then $\underline{C}_{Gr}$ is isomorphic to $\underline{C}$. For every object in $\underline{C}$ is a group object, with the corresponding $\mu: G \times G \to G$ given by the sum of the two canonical projections. On the other hand, every group object structure on $G \in \underline{C}$ must coincide with the given additive one.

   Proof. Suppose $\mu: G \times G \to G$ defines a group object structure on G and let $\alpha * \beta$ denote the corresponding group multiplication in Hom (X,G).

$$X \xrightarrow{(\alpha,\beta)} G \times G \xrightarrow{\mu} G$$

   One obtains $\alpha * \beta = \mu(\alpha,\beta) = \mu((\alpha,o) + (o,\beta)) = \mu(\alpha,o) + \mu(o,\beta) =$ $= \alpha * o + o * \beta = \alpha + \beta$, since it is easily verified that $o_{X,G}$ is the identity element for the operation $*$.
3. $(\mathrm{Top})_{Gr}$ is the category of topological groups, with the continuous homomorphisms as morphisms.
4. In algebraic geometry one studies group objects in the category of schemes.

Chapter III: Adjoint functors and limits.

It is a common situation in various branches of mathematics that one has a pair of categories and functors

$$\underline{C} \underset{T}{\overset{S}{\rightleftarrows}} \underline{D}$$

which are interrelated by natural bijections

$$\eta_{A,B} : \mathrm{Hom}_{\underline{C}} (A,T(B)) \longrightarrow \mathrm{Hom}_{\underline{D}} (S(A), B)$$

for objects $A \in \underline{C}$ and $B \in \underline{D}$. This is what is called an "adjoint" situation, and in this chapter we will study some of its properties and implications.

All categories appearing in this chapter are assumed to be $\underline{U}$-categories.

§ 1. Adjoint functors.

Definition. Let there be given categories and functors

$$\underline{C} \underset{T}{\overset{S}{\rightleftarrows}} \underline{D} \quad .$$

T is said to be a right adjoint of S (and symmetrically S is a left adjoint of T) if there is a natural equivalence

$$\eta : \mathrm{Hom}_{\underline{C}} (\cdot , T(\cdot)) \to \mathrm{Hom}_{\underline{D}} (S(\cdot), \cdot)$$

of bifunctors $\underline{C}^{o} \times \underline{D} \to (\mathrm{Ens})$.

In particular this implies that the functor $\mathrm{Hom}_{\underline{D}} (S(\cdot), B) : \underline{C}^{o} \to (\mathrm{Ens})$ is representable for each $B \in \underline{D}$, with $T(B)$ as the representing object. More precisely we have, as we recall from the theory of representable functors, that $\mathrm{Hom}_{\underline{D}}(S(\cdot), B)$ is represented by a pair $(T(B), \xi_B)$, where $\xi_B \in \mathrm{Hom}(ST(B),B)$ and is obtained by the formula

$$\xi_B = \eta_{T(B),B}(1_{T(B)}) . \qquad (1)$$

Naturality in B implies that we obtain a natural transformation $\xi : ST \to 1_{\underline{D}}$.

We also recall from the general theory that $\eta_{A,B}$ may be reconstructed from ξ_B as

$$\eta_{A,B}(\alpha) = h_B(S(\alpha))(\xi_B) = \xi_B \cdot S(\alpha) \text{ for } \alpha : A \to T(B). \qquad (2)$$

$$\begin{array}{ccc}
\mathrm{Hom}(T(B),T(B)) & \xrightarrow{\eta_{T(B),B}} & \mathrm{Hom}(ST(B),B) \\
\downarrow{\scriptstyle h_{T(B)}(\alpha)} & & \downarrow{\scriptstyle h_B(S(\alpha))} \\
\mathrm{Hom}(A,T(B)) & \xrightarrow[\eta_{A,B}]{} & \mathrm{Hom}(S(A),B)
\end{array}$$

It is worth noting that the naturality of η is an automatic consequence of (2), or more precisely:

<u>Proposition 1</u>. Suppose we are given functors $\underline{C} \underset{T}{\overset{S}{\rightleftarrows}} \underline{D}$ and a natural transformation $\xi : ST \to 1_{\underline{D}}$.
Define

$$\eta_{A,B} : \mathrm{Hom}(A,T(B)) \to \mathrm{Hom}(S(A),B)$$

as $\eta_{A,B}(\alpha) = \xi_B \cdot S(\alpha)$. If $\eta_{A,B}$ is bijective for all A,B, then η makes T into a right adjoint of S.

<u>Proof</u>. We only have to show naturality of $\eta_{A,B}$ in A and B. Let $\gamma : A' \to A$ be a morphim in $\underline{C}$ and consider the diagram

$$\begin{array}{ccc} \mathrm{Hom}(A,T(B)) & \xrightarrow{\eta_{A,B}} & \mathrm{Hom}(S(A),B) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(A',T(B)) & \xrightarrow[\eta_{A',B}]{} & \mathrm{Hom}(S(A'),B) \end{array}$$

Starting with $\alpha : A \to T(B)$ and going the way over the NE corner, we get $\alpha \mapsto \xi_B \cdot S(\alpha) \mapsto \xi_B \cdot S(\alpha) \cdot S(\gamma) = \xi_B \cdot S(\alpha\gamma)$. Going instead over the SW corner we get $\alpha \mapsto \alpha\gamma \mapsto \xi_B \cdot S(\alpha\gamma)$, so the diagram commutes,

To verify naturality in B, let $\beta : B \to B'$ and consider the diagram

$$\begin{array}{ccccc} \mathrm{Hom}(A,T(B)) & \xrightarrow{S} & \mathrm{Hom}(S(A),ST(B)) & \longrightarrow & \mathrm{Hom}(S(A),B) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Hom}(A,T(B')) & \xrightarrow{S} & \mathrm{Hom}(S(A),ST(B')) & \longrightarrow & \mathrm{Hom}(S(A),B') \end{array}$$

where the horizontal arrows in the right square are induced by ξ_B and $\xi_{B'}$. Since ξ is a natural transformation, this square commutes, The left square also commutes since S is a functor. Thus the whole diagram commutes and we have naturality in B,

It follows from the theory of representable functors that a right adjoint of S is uniquely determined (up to natural equivalence). We also remark that representability of $\mathrm{Hom}(S(\cdot),B)$ is sufficient for the existence of a right adjoint of S, indeed:

Proposition 2. The following statements are equivalent for a functor $S : \underline{C} \to \underline{D}$:

(a) S has a right adjoint.

(b) The functor $\mathrm{Hom}_{\underline{D}}(S(\cdot),B)$ is representable for each object B.

Proof. (a) ⇒ (b) is clear. To prove (b) ⇒ (a), choose for each object B in $\underline{D}$ a representing object T(B) with a natural equivalence

$$\varphi_B : \mathrm{Hom}_{\underline{C}}(\cdot, T(B)) \to \mathrm{Hom}_{\underline{D}}(S(\cdot), B).$$

We only have to verify functoriality in B. So let $\beta : B \to B'$ be any morphism in $\underline{D}$. β induces a natural transformation $\psi : \mathrm{Hom}(S(\cdot),B) \to \mathrm{Hom}(S(\cdot),B')$ and thereby gives a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}(\cdot, T(B)) & \xrightarrow{\varphi_B} & \mathrm{Hom}(S(\cdot), B) \\ \downarrow & & \downarrow \psi \\ \mathrm{Hom}(\cdot, T(B')) & \xrightarrow{\varphi_{B'}} & \mathrm{Hom}(S(\cdot), B') \end{array}$$

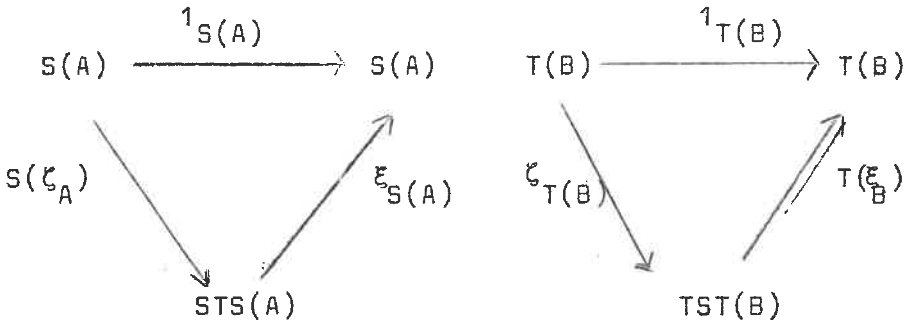
where the left arrow is simply defined as $\varphi_{B'}^{-1}\psi\varphi_B$. By the Yoneda theorem it is induced by a morphism $T(\beta): T(B) \to T(B')$, which is uniquely determined by β. It is easy to see that the unicity implies that T may be considered as a functor, By its construction, T is a right adjoint of S.

The preceding results may be dualized by instead considering S as a left adjoint of T. We then obtain a natural transformation $\zeta: 1_{\underline{C}} \to TS$, defined as $\zeta_A = \eta_{A,S(A)}^{-1}(1_{S(A)})$. η^{-1} may be reconstructed from ζ as

$$\eta_{A,B}^{-1}(\beta) = T(\beta) \cdot \zeta_A \quad \text{for } \beta: S(A) \to B.$$

There is an interesting relation between the two natural transformations $\xi : ST \to 1$ and $\zeta : 1 \to TS$:

Proposition 3. Let T be a right adjoint of $S : \underline{C} \to \underline{D}$. The following two diagrams are commutative:



Proof. By the formula (2) and the definition of ζ we have $\xi_{S(A)}S(\zeta_A) = \eta_{A,S(A)}(\zeta_A) = 1_{S(A)}$. Commutativity of the second diagram follows by duality.

Proposition 4. Let $\underline{C}$ and $\underline{D}$ be preadditive categories and let T be a right adjoint of $S : \underline{C} \to \underline{D}$. Then T is an additive functor if and only if S is additive.

Proof. We will show that S is additive if and only if $\eta_{A,B}$ is a group isomorphism. Then by symmetry we will have that T is additive if and only if $\eta_{A,B}^{-1}$ is a group isomorphism, and our assertion will be proved. If S is additive, then for $\alpha_1, \alpha_2 : A \to T(B)$ we get $\eta(\alpha_1 + \alpha_2) = \xi_B S(\alpha_1 + \alpha_2) =$
$= \xi_B S(\alpha_1) + \xi_B S(\alpha_2) = \eta(\alpha_1) + \eta(\alpha_2)$.

Conversely, assume $\eta_{A,B}$ is additive. For a morphism $\gamma: A' \to A$ we have $\eta(\zeta_A\gamma) = \xi_{S(A)}S(\zeta_A\gamma) = \xi_{S(A)}S(\zeta_A)S(\gamma) = S(\gamma)$ by (2) and Prop. 3. It follows that additivity of η implies additivity of S.

Examples:

1. Let A and B be two rings and consider modules L_A, ${}_AM_B$ and N_B (i.e. L is a right A-module, M is an A-B-bimodule and N is a right B-module). The tensor product $L \otimes_A M$ becomes a right B-module by putting $(x \otimes y)b = x \otimes yb$ for $b \in B$. Similarly $\mathrm{Hom}_B(M,N)$ becomes a right A-module by defining

$\varphi a(x) = \varphi(ax)$ for $a \in A$, $x \in M$ and $\varphi : M \to N$. We have a natural equivalence of functors

$$\mathrm{Hom}_B\ (L \otimes_A M,\ N) \xrightarrow{\cong} \mathrm{Hom}_A\ (L,\ \mathrm{Hom}_B(M,N))$$

by ([17], p. 144). This means precisely that $\otimes_A M : \mathrm{Mod}\ (A). \to \mathrm{Mod}\ (B).$ is a left adjoint of $\mathrm{Hom}_B\ (M,\ \cdot) : \mathrm{Mod}\ (B). \to \mathrm{Mod}\ (A).$.

2. Let $\varphi : A \to B$ be a ring homomorphism. We may consider B as an A-B-bimodule by defining $ab = \varphi(a)b$ for $a \in A$, $b \in B$, But B may equally well be considered as a B-A-bimodule. Define functors

$$\mathrm{Mod}\ (A).\ \underset{\varphi^{!}}{\overset{\varphi^{*}}{\underset{\longleftarrow\ \varphi_* \ \longrightarrow}{\longrightarrow}}}\ \mathrm{Mod}\ (B).$$

as

$\varphi^*(M) = M \otimes_A B$ (extension of scalars)

$\varphi_*(N) = N$ (restriction of scalars: $xa = x\varphi(a)$ for $a \in A$, $x \in N$)

$\varphi^!(M) = \mathrm{Hom}_A(B,M)$ (where B is a B-A-bimodule).

φ^* is then a left adjoint of φ_* while $\varphi^!$ is easily verified to be a right adjoint of φ_* .

3. Consider the functors $(Ab) \underset{F}{\overset{G}{\rightleftarrows}} (Ens)$, where G is the forgetful functor and F is the construction of free abelian groups on sets (ch. 2, §1). G is a right adjoint of F.

4. The loop space functor $\Omega : (Top)_o \to (Top)_o$ is a right adjoint of the suspension functor ([21], p. 41).

Exercises:

1. Let T be a right adjoint of $S: \underline{C} \to \underline{D}$. Show that T is faithful if and only if $\xi_B : ST(B) \to B$ is an epimorphism for every B.

2. Suppose we have functors $\underline{C} \underset{T}{\overset{S}{\rightleftarrows}} \underline{D}$ and natural transformations $\xi : ST \to 1$ and $\zeta : 1 \to TS$. Show that if the two diagrams displayed in Prop. 3 are commutative, then T is a right adjoint of S.

§ 2. Equivalences.

An equivalence was defined in ch. 2 as a functor $S: \underline{C} \to \underline{D}$ which is full and faithful and such that every $B \in \underline{D}$ is isomorphic to some $S(A)$. This definition is not very satisfactory since it lacks in symmetry, but it has certain computational advantages. The following result gives a more complete picture of the properties of an equivalence.

Proposition 5. The following statements are equivalent for a functor $S : \underline{C} \to \underline{D}$:

(a) S is an equivalence (in the sense above).

(b) There exist $T : \underline{D} \to \underline{C}$ and natural equivalences $1_{\underline{C}} \to TS$ and $1_{\underline{D}} \to ST$.

(c) There exists $T : \underline{D} \to \underline{C}$ which is both a left and a right adjoint of S, and such that the canonical natural transformations $1_{\underline{C}} \to TS$ and $1_{\underline{D}} \to ST$ are natural equivalences.

<u>Proof</u>. (c) $\Rightarrow$ (b) trivially.

(b) $\Rightarrow$ (a): The natural equivalence $1 \to TS$ gives for every morphism $A \to A'$ in $\underline{C}$ a commutative diagram

$$\text{(1)} \qquad \begin{array}{ccc} A & \xrightarrow{\cong} & TS(A) \\ \downarrow & & \downarrow \\ A' & \xrightarrow{\cong} & TS(A') \end{array}$$

which shows that S is faithful. By symmetry also T is faithful. A morphism β: $S(A) \to S(A')$ induces conversely α: $A \to A'$ so that (1) commutes, But then also $TS(\alpha) = T(\beta)$, and $\beta = S(\alpha)$ since T is faithful. Thus S is full. Finally there exists for every object $B \in D$ an isomorphism $B \cong ST(B)$. Hence S is an equivalence.
(a) $\Rightarrow$ (c): If S is an equivalence, we can for each $B \in \underline{D}$ find an object $T(B) \in \underline{C}$ and an isomorphism ξ_B : $ST(B) \to B$. A morphism β: $B \to B'$ in $\underline{D}$ induces a morphism $\xi_{B'}^{-1}\beta\,\xi_B$: $ST(B) \to ST(B')$, and since S is full and faithful there is a unique morphism $T(B) \to T(B')$, denoted by $T(\beta)$, such that $\xi_{B'}^{-1}\beta\xi_B = ST(\beta)$. We thus have the commutative diagram

$$\text{(2)} \qquad \begin{array}{ccc} ST(B) & \xrightarrow{\xi_B} & B \\ {\scriptstyle ST(\beta)}\downarrow & & \downarrow{\scriptstyle \beta} \\ ST(B') & \xrightarrow[\xi_{B'}]{} & B' \end{array}$$

It is easy to see that T in this way becomes a functor, and ξ : $ST \to 1$ a natural equivalence.

For each $A \in \underline{C}$ we get in particular an isomorphism $\xi_{S(A)}$: $STS(A) \to S(A)$, and since S is full and faithful we may write $\xi_{S(A)} = S(\zeta_A^{-1})$ for a unique isomorphism ζ_A : $A \to TS(A)$.

To prove that ζ_A is natural in A, consider the diagram

$$(3) \qquad \begin{array}{ccc} A & \xrightarrow{\ \zeta_A\ } & TS(A) \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle TS(\alpha)} \\ A' & \xrightarrow[\ \zeta_{A'}\ ]{} & TS(A') \end{array}$$

for any morphism α in $\underline{C}$. Applying S to this diagram we get a commutative diagram by the definition of ζ and naturality of ξ. Since S is faithful, (3) must then be commutative. Hence $\zeta : 1 \to TS$ is a natural equivalence.

To prove that T is a right adjoint of S we use Prop. 1 and define

$$\eta_{A,B} : \text{Hom}\ (A,T(B)) \to \text{Hom}\ (S(A),B)$$

as $\eta_{A,B}\ (\alpha) = \xi_B\ S(\alpha)$. Since ξ_B is an isomorphism and S is full and faithful, $\eta_{A,B}$ will be bijective. So T is a right adjoint of S. To prove that T is a left adjoint of S we similarly apply Prop. 1* to $\xi^{-1} : 1 \to ST$.

§ 3. Limits and colimits.

Before introducing the quite general notions of limits and colimits, we should recall the "classical" definitions of inverse and direct limits of sets.

A <u>direct system</u> of sets is a family (A_i) of sets, indexed by a partially ordered set I which is directed (i.e. for every pair i, j in I, there exists $k \in I$ such that $i \leq k$, $j \leq k$), and maps $\alpha_{ij} : A_i \to A_j$ whenever $i \leq j$, such that

$$\alpha_{ii} = \text{id.}\ ,$$

$$\alpha_{jk}\ \alpha_{ij} = \alpha_{ik} \quad \text{when } i \leq j \leq k.$$

The <u>direct limit</u> of this system is defined as $\varinjlim A_i = \coprod_I A_i/\sim$ where $\sim$ identifies two elements if they "finally coincide", i.e. if $x \in A_i$ and $y \in A_j$, then $x \sim y$ if there exists k such that $i \leq k$, $j \leq k$ and $\alpha_{ik}(x) = \alpha_{jk}(y)$. Note that $\sim$ really is an equivalence relation.

An <u>inverse system</u> is obtained just by reversing the arrows, i.e. $\alpha_{ij} : A_j \to A_i$ whenever $i \leq j$. The <u>inverse limit</u> of such a system describes the "far past" of elements and is a subset of ΠA_i, namely $\varprojlim A_i = \{(x_i)_I \mid \text{if } i \leq j, \text{ then } \alpha_{ij}(x_j) = x_i\} \subset \prod_I A_i$. For more details about direct and inverse limits of sets, spaces or groups, see [8], Ch. 8.

The inverse limit $\varprojlim A_i$ has the following universal mapping property (which in fact characterizes it uniquely up to isomorphisms): there is a canonical family of maps $\xi_i : \varprojlim A_i \to A_i$ which is compatible, i.e. if $i \leq j$ then $\alpha_{ij}\,\xi_j = \xi_i$, and for any object B and compatible family of maps $\psi_i : B \to A_i$, there exists a unique map $\beta : B \to \varprojlim A_i$ such that $\xi_i\,\beta = \psi_i$ for all i. There is a similar characterization of direct limits. These universal mapping properties should of course be used when defining limits and colimites in the general case.

Let $\underline{C}$ be a $\underline{U}$-category (as usual) and let $\underline{I}$ be a $\underline{U}$-small category (which might be thought of as an "index category"). We want to define the limit for functors $\underline{I} \to \underline{C}$. As a first step we introduce a functor $k : \underline{C} \to \text{Fun}(\underline{I},\underline{C})$ as:

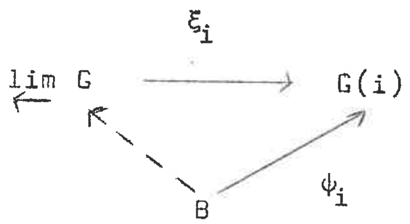
> if C is an object of $\underline{C}$, then $k_C : \underline{I} \to \underline{C}$ is defined to be the constant functor given by
>
> $k_C(i) = C$ for $i \in \text{Ob}(\underline{I})$
>
> $k_C(\lambda) = 1_C$ for $i \in \text{Mor}(\underline{I})$;
>
> if $\alpha : C \to D$ is a morphism in $\underline{C}$, then $k(\alpha) : k_C \to k_D$ is the obvious natural transformation, i.e. $k(\alpha)_i = \alpha$ for all $i \in \text{Ob}(\underline{I})$.

We now define the <u>limit</u> functor $\varprojlim : \text{Fun}(\underline{I},\underline{C}) \to \underline{C}$ as being a right adjoint of k. So if $G : \underline{I} \to \underline{C}$ is any functor, then its <u>limit</u> (sometimes called its <u>projective limit</u>) $\varprojlim G$ is determined up to isomorphisms by the formula

$$\mathrm{Hom}_{\underline{C}}\,(C, \varprojlim G) \cong \mathrm{Nat}\,(k_C, G).$$

This may also be expressed by saying that $\varprojlim G$ represents the functor $C \mapsto \mathrm{Nat}\,(k_C, G)$. Recalling the characterization of representing pairs given in Ch. 2, § 5, we may describe limits in the following explicit fashion:

given $G : \underline{I} \to \underline{C}$, a limit for G is an object $\varprojlim G$ in $\underline{C}$ together with a family of morphisms $\xi_i : \varprojlim G \to G(i)$ which is compatible, i.e. for every $\lambda : i \to j$ in $\underline{I}$ one has $G(\lambda)\xi_i = \xi_j$, such that each compatible family $\psi_i : B \to G(I)$ may be uniquely factored over $(\xi_i)_{\underline{I}}$.



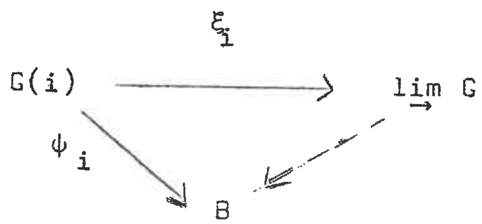
If the limit functor $\varprojlim : \mathrm{Fun}\,(\underline{I}, \underline{C}) \to \underline{C}$ exists, then $\underline{C}$ is said to "have $\underline{I}$-limits". Similarly $\underline{C}$ is said to "have finite limits" if it has $\underline{I}$-limits for all finite $\underline{I}$, and $\underline{C}$ is said to be $\underline{U}$-complete if it has $\underline{I}$-limits for all $\underline{U}$-small categories $\underline{I}$.

If $\underline{C}$ is a preadditive category, then it follows from Prop. 4 that $\varprojlim$ is an additive functor, since k obviously is additive.

The colimit functor $\varinjlim : \mathrm{Fun}\,(\underline{I}, \underline{C}) \to \underline{C}$ is obtained by dualization as being a left adjoint of k, We consequently have

$$\mathrm{Hom}_{\underline{C}}\,(\varinjlim G, C) \cong \mathrm{Nat}\,(G, k_C)$$

and a corresponding explicit description of $\varinjlim G$, which is indicated by the diagram

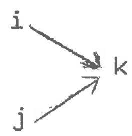


Examples:

1. Let $\underline{I}$ be a discrete category, i.e. the only morphisms are the identities. If $G : \underline{I} \to \underline{C}$, then $\varprojlim G = \prod_{Ob\,\underline{I}} G(i)$ and $\varinjlim G = \coprod_{Ob\,\underline{I}} G(i)$.

2. Let $\underline{I}$ be a directed category, i.e. a partially ordered set such that for each pair i, j there exists k with $i \leq k$, $j \leq k$. A functor $G : \underline{I} \to \underline{C}$ is called a <u>direct system</u> in $\underline{C}$ and $\varinjlim G$ is a <u>direct limit</u>, while a functor $G : \underline{I}^{o} \to \underline{C}$ is an <u>inverse system</u> in $\underline{C}$ and $\varprojlim G$ is an <u>inverse limit</u>.

3. Let $\underline{I}$ be the directed category



(+ identities)

and $G : \underline{I} \to \underline{C}$. Then $\varprojlim G = G(i) \times_{G(k)} G(j)$ and $\varinjlim G = G(k)$.

<u>Proposition 6</u>. A $\underline{U}$-category $\underline{C}$ is $\underline{U}$-complete if and only if $\underline{C}$ has equalizers and $\underline{U}$-products.

<u>Proof</u>. Since equalizers are particular instances of pullbacks, it follows from the preceding examples that if $\underline{C}$ is $\underline{U}$-complete, then it has equalizers and $\underline{U}$-products.

The converse follows from the fact that if $\underline{I} \in \underline{U}$ and $G : \underline{I} \to \underline{C}$, then the limit of G is given by the formula

$$\varprojlim G = \mathrm{Equ}\left(\prod_{i \in Ob\,\underline{I}} G(i) \underset{\psi}{\overset{\varphi}{\rightrightarrows}} \prod_{\lambda \in I} G\,(\mathrm{target}\ \lambda\ ) \right) \qquad (1)$$

(If $\lambda : i \to j$ is a morphism in $\underline{I}$, then j is called the <u>target</u> of λ while i is the <u>source</u> of λ). φ is defined by the canonical projections $\prod_{Ob\,\underline{I}} G(i) \to G(\mathrm{target}\ \lambda\ )$, using the universal property of the second $\prod$. ψ is similarly defined by the compositions

$$\prod_{\mathrm{Ob}\,\underline{I}} G(i) \xrightarrow[\text{proj.}]{} G\,(\text{source } \lambda\,) \xrightarrow[G(\lambda)]{} G\,(\text{target } \lambda\,).$$

It is easy to see that the formula above really gives the limit: define ξ_j : Equ $(\ldots) \to \prod G(i) \to G(j)$ by composing the canonical morphisms. The family (ξ_j) is clearly compatible, and it is easy to verify that it has the required universal property.

The dual formula for the colimit is:

$$\varinjlim G = \mathrm{Coequ}\left(\coprod_{\lambda \in \underline{I}} G\,(\text{source } \lambda\,) \;\underset{\psi}{\overset{\varphi}{\rightrightarrows}}\; \coprod_{i \in \mathrm{Ob}\underline{I}} G(i) \right) \qquad (2)$$

with φ defined by canonical injections $G(\text{source } \lambda\,) \rightarrowtail \coprod G(i)$, and ψ defined by $G\,(\text{source } \lambda\,) \xrightarrow[G(\lambda)]{} G\,(\text{target } \lambda\,) \xrightarrow[\text{inj.}]{} \coprod G(i)$.

In particular we have that the categories $\underline{U}$ - (Ens) and $\underline{U}$ - (Ab) are $\underline{U}$-complete and $\underline{U}$-cocomplete. Limits are for both categories given by

$$\varprojlim G = \{ (x_i) \in \prod G(i) \mid G(\lambda)(x_i) = x_j \text{ for each } \lambda : i \to j \};$$

The explicit formulas for colimits are somewhat complicated when considered over arbitrary $\underline{I}$. We will therefore impose extra conditions on $\underline{I}$.

<u>Definition</u>. A category $\underline{I}$ is called <u>pseudo-directed</u> if it satisfies:

PD 1. For any objects i,j there exists a diagram

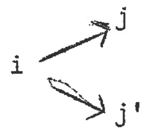
$$\begin{array}{ccc} i & \searrow & \\ & & k \\ j & \nearrow & \end{array}$$

PD 2. For every diagram $i \underset{\mu}{\overset{\lambda}{\rightrightarrows}} j$ there exists $\eta : j \to k$ such that $\eta\lambda = \eta\mu$.

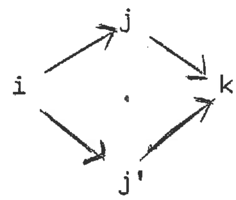
We extend the classical terminology somewhat by calling every limit (colimit) over a pseudo-directed category an <u>inverse (direct) limit</u>. As it turns out, this is really no genuine generalization of the classical notions (appendix A).

A pseudo-directed category $\underline{I}$ satisfies the following weaker axiom:

PD 3. Every diagram



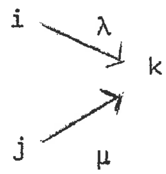
may be inserted in a commutative diagram



If $\underline{I}$ satifies PD 3 and $G : \underline{I} \to (Ens)$, then

$$\varinjlim G = \coprod_{Ob\underline{I}} G(i) / \sim$$

where $x \sim y$ ($x \in G(i)$, $y \in G(j)$) if and only if there exists a diagram



with $G(\lambda)(x) = G(\mu)(y)$.

<u>Proof</u>. The relation $\sim$ is an equivalence relation: it is trivially reflexive and symmetric, and it is easy to see that PD3 makes it transitive. The family of composed maps

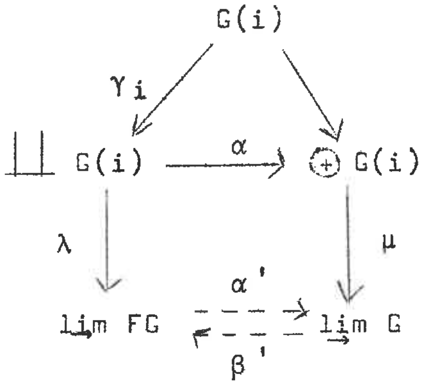
$$G(i) \to \coprod G(i) \to \coprod G(i)/\sim$$

is clearly compatible, and one verifies its universality.

If $\underline{I}$ does not satisfy PD3, one has to define $\sim$ in a more intricate way to make it an equivalence relation (see [1]). If $\underline{I}$ is pseudo-directed, the formula above also gives colimits for group valued systems:

Proposition 7. Let $G : \underline{I} \to (Ab)$ and suppose $\underline{I}$ is pseudo-directed. Let $F : (Ab) \to (Ens)$ be the forgetful functor. Then $F(\varinjlim G) = \varinjlim FG$.

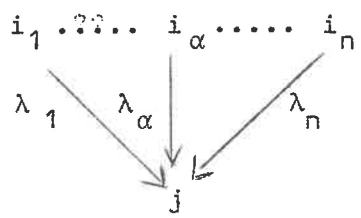
Proof. Consider the diagram



where the objects in the left column are in (Ens) and those in the right column are in (Ab), and where α is the canonical map. The compatible family of maps $\mu\alpha\gamma_i$ induces a map $\alpha' : \varinjlim FG \to \varinjlim G$. We assert that α' is surjective. To see this it suffices to show that $\mu\alpha$ is surjective. Now we have

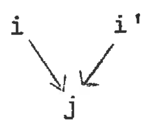
$$\varinjlim G = \bigoplus_i G(i) \,/\, \mathrm{Im}\,(\varphi - \psi)$$

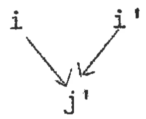
with φ and ψ defined as in formula (2). An element $x \in \varinjlim G$ is thus represented by a finite sum $\sum_{\alpha=1}^{n} x_{i_\alpha} \in \bigoplus G(i)$. Using PD 1 we may find a diagram



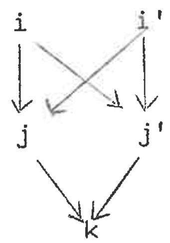
in $\underline{I}$, and we put $y_\alpha = G(\lambda_\alpha)(x_{i_\alpha}) \in G(j)$ and $y = \Sigma\, y_\alpha$. We obtain $(\varphi - \psi)\, \Sigma\, x_{i_\alpha} = \Sigma\, x_{i_\alpha} - \Sigma\, y_\alpha = \Sigma\, x_{i_\alpha} - y$, so in $\varinjlim G$ we have $x = \bar{y} \in \mathrm{Im}\, \mu\alpha$.

We will then define a group structure on the set $\varinjlim FG$ so that each $\lambda\gamma_i$ becomes a group homomorphism. Let $\bar{x}, \bar{y} \in \varinjlim FG$ be represented by $x, y \in G(i)$ (PD1 is used here). Put $\bar{x} + \bar{y} = \overline{x + y}$. We only have to verify that this is well-defined, for the group axioms are then clearly satisfied and $\lambda\gamma_i$ are homomorphisms. So suppose $\bar{x}$ and $\bar{y}$ also are represented by x' and y' in $G(i')$. There exist diagrams





such that x and x' have the same image in $G(j)$, while y and y' have the same image in $G(j')$. Because of PD3 there exist $j \to k$, $j' \to k$ such that the left side square of



commutes. By PD2, k may be chosen so that also the right hand square commutes. Then x and x' (resp. y and y') have the same image in $G(k)$, and $\overline{x + y} = \overline{x' + y'}$.

Now the compatible family of group homomorphisms $\lambda\gamma_i$ induces a homomorphism $\beta' : \varinjlim G \to \varinjlim FG$ such that $\beta'\mu\alpha\gamma_i = \lambda\gamma_i$. Then $\beta'\alpha'\lambda\gamma_i = \beta'\mu\alpha\gamma_i = \lambda\gamma_i$, and by unicity we obtain $\beta'\alpha' = 1$. Since we have already seen that α' is surjective, it follows that α' is bijective.

Limits in functor categories:

Our purpose is to show that if $\underline{C}$ is $\underline{U}$-small and $\underline{D}$ is a $\underline{U}$-complete $\underline{U}$-category, then Fun $(\underline{D},\underline{D})$ is also $\underline{U}$-complete and its limits may be computed "point-wise". Let $G : \underline{I} \to$ Fun $(\underline{C},\underline{D})$, where $\underline{I}$ is $\underline{U}$-small. For each object A of $\underline{C}$, define a functor $G_A : \underline{I} \to \underline{D}$ as

$$G_A(i) = G(i)\ (A)$$

$$G_A(\lambda) = G(\lambda)_A \qquad \text{(verify functoriality!).}$$

A morphism $\alpha : A \to B$ in $\underline{C}$ induces a natural transformation $\bar{\alpha} = G_A \to G_B$ as $\bar{\alpha}_i = G(i)\ (\alpha) : G_A(i) \to G_B(i)$.
We assert that $\varprojlim G$ is the functor $\underline{C} \to \underline{D}$ given as:

$$\varprojlim G\ (A) = \varprojlim G_A \qquad \text{for each } A,$$

$$\varprojlim G((\alpha)) = \varprojlim \bar{\alpha} \qquad \text{for } \alpha : A \to B.$$

To verify this we must show that if $F : \underline{C} \to \underline{D}$ is defined as $F(A) = \varprojlim G_A$, $F(\alpha) = \lim \bar{\alpha}$, then F is a functor (which is rather evident) and the family of natural transformations $\{F \to G(i)\}_{i \in \underline{I}}$ is compatible and has the necessary universal property. But this is quite clear since it holds for each object A of $\underline{C}$. The conclusion is that limits may be computed "point-wise"; in special cases this has been noted already in Ch. II, § 4 (products, kernels etc.). The result may be explicitly formulated as:

<u>Proposition 8.</u> Let $\underline{I}$ and $\underline{C}$ be $\underline{U}$-small categories. If $\underline{D}$ is a $\underline{U}$-complete $\underline{U}$-category, then so is Fun($\underline{C}$, $\underline{D}$). If $G : \underline{I} \to \text{Fun}(\underline{C}, \underline{D})$, then $\varprojlim G$ is the composed functor

$$\underline{C} \xrightarrow{G.} \text{Fun}\ (\underline{I}, \underline{D}) \xrightarrow{\varprojlim} \underline{D}$$

where $G.\ (A) = G_A$
$G.\ (\alpha) = \bar{\alpha}$.

If $\underline{C}$ and $\underline{D}$ are preadditive, a similar result holds for <u>Hom</u> ($\underline{C}$, $\underline{D}$).

This result may in particular be applied to the categories Fun ($\underline{C}$, $\underline{U}$ - (Ens)) and <u>Hom</u> ($\underline{C}$, $\underline{U}$ - (Ab)).

Exercises:

1. Let $G : \underline{I} \to \underline{C}$. Show that there exist natural bijections:

$$\mathrm{Hom}\,(C, \varprojlim G) \cong \varprojlim \mathrm{Hom}\,(C, G(\cdot)),$$

$$\mathrm{Hom}\,(\varinjlim G, C) \cong \varprojlim \mathrm{Hom}\,(G(\cdot), C)$$

(cf. ch. 1, § 4, ex. 2).

2. Show that if $(C_i)_I$ is a family of objects, then $\coprod_I C_i$ is the direct limit of the coproducts $\coprod_J C_i$ for finite subsets J of I. (In particular, if $\underline{C}$ is abelian, then $\underline{C}$ is cocomplete if and only if it has direct limits).

§ 4. Preservation properties of limits.

Let $T : \underline{C} \to \underline{D}$ be a functor. T is said to preserve (or commute with) limits if for every $G : \underline{I} \to \underline{C}$ such that $\varprojlim G$ exists, one has $T(\varprojlim G) = \varprojlim TG$ (it is then understood that if $\xi_i : \varprojlim G \to G(i)$ is the canonical morphism, then $T(\xi_i)$ should be the canonical morphism $\varprojlim TG \to TG(i)$). There is a similar terminology in case T only preserves special classes of limits, e.g. kernels, products, finite limits or inverse limits. Dually for colimits.

Example: The forgetful functor (Ab) → (Ens) preserves limits and pseudo-directed colimits (Prop. 7).

Proposition 9. Suppose $\underline{C}$ is $\underline{U}$-complete. $T : \underline{C} \to \underline{D}$ preserves $\underline{U}$-limits if and only if it preserves equalizers and $\underline{U}$-products.

Proof. Immediate from the formula for limits given in the proof of Prop. 6.

Proposition 10. Suppose $\underline{C}$ and $\underline{D}$ are additive categories and $T : \underline{C} \to \underline{D}$ is a functor. Then :

(i) T is additive if and only if it preserves finite products.

(ii) T preserves finite limits if and only if it is additive and preserves kernels.

Proof. (i): Same as for ch. 2, Prop. 1. (ii) follows then immediately.

From the definition of limit and colimit we obtain the following basic result:

Proposition 11. A right adjoint functor preserves limits, while a left adjoint functor preserves colimits.

Proof. Let T be a right adjoint of $S : \underline{C} \to \underline{D}$, and suppose $G : \underline{I} \to \underline{D}$ is a functor such that $\varprojlim G$ exists. For any abject C of $\underline{C}$ we then have $\mathrm{Hom}_{\underline{C}}\,(C, T(\varprojlim G)) \cong \mathrm{Hom}_{\underline{D}}\,(S(C), \varprojlim G) \cong$ $\cong \mathrm{Nat}\,(k_{S(C)}, G) \cong \mathrm{Nat}\,(k_C, TG)$ as is easily verified. By the definition of limits, this means that $\varprojlim TG$ exists and that in fact $T\,(\varprojlim G) = \varprojlim TG$. Dually for left adjoints.

Corollary 1. The functor $\varprojlim : \mathrm{Fun}\,(\underline{I}, \underline{C}) \to \underline{C}$ preserves limits, while the corresponding functor $\varinjlim$ preserves colimits,

Corollary 2. Suppose $\underline{C}$ and $\underline{D}$ are additive categories and the functor $T : \underline{C} \to \underline{D}$ is either a right or a left adjoint. T is then necessarily additive.

Proof. Use Prop. 10 (i).

Corollary 3. An equivalence preserves limits and colimits.

Examples:

1. Since the tensor product is a left adjoint, we infer from Prop. 11 that it is right exact and preserves direct limits.

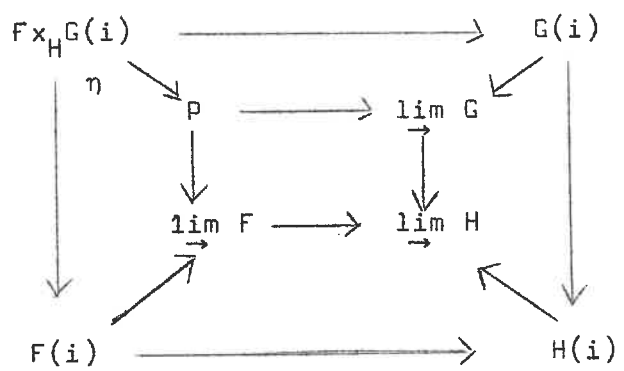
2. The forgetful functor (Ab) → (Ens) is a right adjoint and therefore preserves limits (as was noted already in the preceding §).

3. The functor $h^B : \underline{C} \to (\mathrm{Ens})$ defined as $h^B(C) = \mathrm{Hom}\,(B,C)$ (ch. 2, § 1) preserves limits (exercise 1 in the preceding §). Similarly $h_B : \underline{C}^o \to (\mathrm{Ens})$ defined as $h_B(C) = \mathrm{Hom}\,(C,B)$ preserves limits, i.e. takes colimits in $\underline{C}$ to limits in (Ens).

Arbitrary limits or colimits are obviously in general not exact, but we have:

<u>Theorem 12.</u> If $\underline{I}$ is pseudo-directed, then $\varinjlim : \mathrm{Fun}(\underline{I},(\mathrm{Ens})) \to (\mathrm{Ens})$ and $\varinjlim : \mathrm{Fun}(\underline{I},(\mathrm{Ab})) \to (\mathrm{Ab})$ preserve finite limits.
(This is usually stated as: Pseudo-directed colimits are exact in (Ens) and (Ab)).

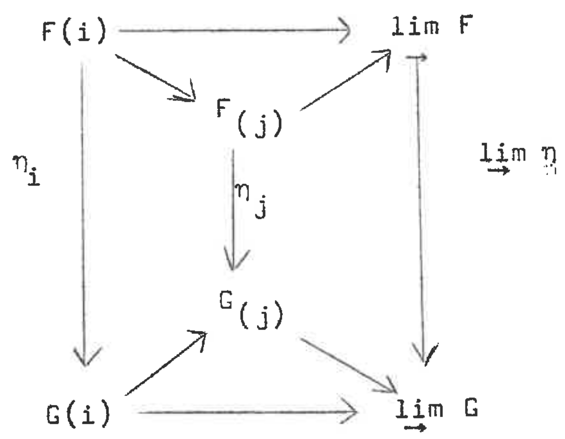
<u>Proof</u>. It suffices to show that $\varinjlim$ preserves pullbacks. So let F, G, H be three functors $\underline{I} \to (\mathrm{Ens})$ with a pullback $F \times_H G$. Write $P = \varinjlim F \times_{\varinjlim H} \varinjlim G$. For each $i \in \underline{I}$ we have a diagram



where both the inner and the outer square are pullback diagrams. η_i is induced by the pullback property of the inner square.
In order to show that $P = \varinjlim Fx_HG$, consider any compatible family $\psi_i : Fx_HG(i) \to B$ $(i \in \underline{I})$ and show its unique factorization over (η_i). So we try to define a morphism $\psi : P \to B$ in the following way. If $x \in P$, then $x = (\bar{x}_i, \bar{y}_j) \in \varinjlim F \times \varinjlim G$ with $\bar{x}_i$ represented by $x_i \in F(i)$, $\bar{y}_j$ by $y_j \in G(j)$. By PD1 we may assume $i = j$. Since $\bar{x}_i$ and $\bar{y}_i$ have the same image in $\varinjlim H$, there exist two arrows $\lambda, \mu : i \to i'$ in $\underline{I}$ such that $H(\lambda)x = H(\mu)y$. Using PD2 we may therefore write x as $x = (\bar{x}_k, \bar{y}_k)$

with $(x_k, y_k) \in F(k) \times_{H(k)} G(k)$ for some $k \in \underline{I}$. We now put $\phi(x) = \phi_k(x_k, y_k)$. ϕ is well-defined becuase the family (ϕ_i) is compatible, and it is also clear that ϕ is the unique morphism such that $\phi\eta_i = \phi_i$.

We now consider the (Ab) case. Since we already know that $\varinjlim$ is right exact, it suffices to consider preservation of monomorphisms. Let $\eta : F \to G$ be a monomorphism in the functor category and suppose $x \in \varinjlim F$ is mapped to zero by $\varinjlim \eta : \varinjlim F \to \varinjlim G$. x is represented by some $x_i \in F(i)$ by Prop. 7. $\eta_i(x_i)$ maps canonically to zero in $\varinjlim G$, so there exists $\lambda : i \to j$ such that $G(\lambda)(\eta_i(x_i)) = 0$. But $G(\lambda)\eta_i = \eta_j F(\lambda)$, so $F(\lambda) x_i = 0$. It follows that $x = 0$.



Corollary. Pseudo-directed colimits are exact in Mod (A)..

Proof. The proof for (Ab) works also in Mod(A)., or alternatively use the exercise below.

Exercises:

1. Show that if $\underline{C}$ is $\underline{U}$-small, and $\underline{D}$ is an $\underline{I}$-complete abelian $\underline{U}$-category where $\underline{I}$-limits are exact, then Fun $(\underline{C}, \underline{D})$ has exact $\underline{I}$-limits. Similarly for $\underline{\text{Hom}}$ $(\underline{C}, \underline{D})$ when $\underline{C}$ is preadditive.

2. Show that Th. 12 in the (Ab) case may be generalized to $\underline{I}$ satisfying only PD2 and PD3. (Hint: write $\underline{I}$ is a disjoint union of pseudo-directed categories).

Appendix III. A. Pseudo-directed categories.

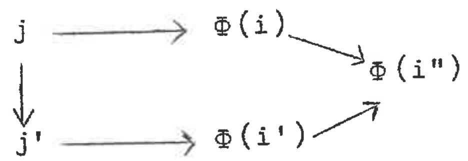
Definition. Let $\underline{I}$ and $\underline{J}$ be pseudo-directed categories. A functor $\Phi : \underline{I} \to \underline{J}$ is cofinal (although "final" would be more appropriate) if

1. For every $j \in J$ there exists $i \in I$ and a morphism $j \to \Phi(i)$.
2. If $j \in J$ and $i \in I$, and there are two morphisms $j \rightrightarrows \Phi(i)$, then there exists a morphism $i \to i'$ in $\underline{I}$ such that the composed morphisms $j \rightrightarrows \Phi(i')$ are equal.

The reason for studying cofinal functors is the following result:

Proposition 1. Let $\underline{I}$ and $\underline{J}$ be $\underline{U}$-small pseudo-directed categories, and $\Phi : \underline{I} \to \underline{J}$ cofinal. If $F : \underline{J} \to \underline{C}$ is a functor where $\underline{C}$ is a cocomplete $\underline{U}$-category, then the canonical morphism $\varinjlim_{\underline{I}} F\Phi \to \varinjlim_{\underline{J}} F$ is an isomorphism.

Proof. We construct an inverse morphism $\varinjlim_{J} F \to \varinjlim_{I} F\Phi$ in the following way. For $j \in J$ we choose a morphism $j \to \Phi(i)$ by using (1) for Φ. This gives a morphism $F(j) \to F\,\Phi(i) \to \varinjlim_{I} F\,\Phi$. We have to verify that this family is compatible. If $j \to j'$ in $\underline{J}$, then we get a diagram

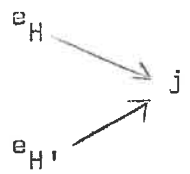


where i" exists by PD1 for $\underline{I}$. By using (2) for Φ, we may assume the diagram commutative by taking i" big enough. So the family $F(j) \to \varinjlim_{I} F\Phi$ is compatible, and hence induces a morphism $\varinjlim_{J} F \to \varinjlim_{I} F\Phi$. It is an easy exercise to show that this morphism is the inverse of the canonical morphism.

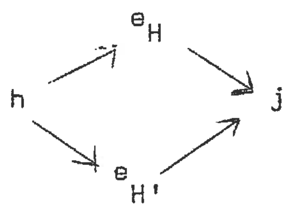
Note that by duality we get a similar result for inverse limits.

Proposition 2. (P. Deligne) Let J be a U-small pseudo-directed category. Then there exists a U-small directed category I and a cofinal functor I → J.

Proof. Let I the set of finite subcategories of J having a unique final object, and let it be partially ordered by inclusion. I is certainly U-small, and we will verify that it is directed. Let H and H' be objects of I, with final objects e_H and $e_{H'}$. By PD1 there exists $j \in J$ and a diagram

(*) 

Let H" be the subcategory of J obtained by taking the union of H, H' and the diagram (*) . In H" there may exist diagrams of the type



for $h \in H \cap H'$. Since there exist only a finite number of such diagrams, we may use PD2 and take j big enough to become a final object in H".

I is thus directed. Let $\Phi : I \to J$ be the functor given by $H \mapsto e_H$. Φ is obviously cofinal.

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B. Stenström: Categorical Algebra.

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Page

15 Exercise 3 should read: Show that the category of abelian torsion groups has coproducts and products.

17 Exercise 2: read "products" instead of "intersections".

20 The first line should read: Hence the notions of finite product and coproduct ....

28 In the proof of Prop. 3, the fact that α'' is a monomorphism does not follow from the 5 lemma. Instead one has to argue as follows: let $K = \operatorname{Ker} \alpha''$ and consider the pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & K \\ {\scriptstyle\beta}\downarrow & & \downarrow \\ A_1 & \longrightarrow & A_1/A_1 \cap A_2 \end{array}$$

Since $\alpha\beta$ goes to zero in $A_1 \cup A_2/A_2$, it factors over A_2. The left square of the diagram on p. 28 is a pullback, so it follows that β may be factored over $A_1 \cap A_2$. But then $P \to K \to A_1/A_1 \cap A_2$ is zero, and since $P \to K$ is an epimorphism (by the preceding appendix), we may conclude that $K = 0$.

51 Line 11: to make $\coprod G_i$ a generator, we should also assume that all $\operatorname{Hom}(G_i, A) \neq \emptyset$ for all $i \in I$ and all A.

73 The arrow $F \times_H G(1) \to F$ in the diagram is called η_1 and not η.