Compact Riemann Surfaces

Introduction

Everything in these notes is contained in R.C. Gunning's Lectures on Riemann Surfaces (Princeton Mathematical Notes 1966). My aim has only been to illuminate a part of the material from this book and the reader is therefore urged to study Gunning's book for more details, and of course we have not touched the whole material. Roughly speaking we cover page 1 - 110 and Abel's Theorem , as stated on page 162 as Corollary 3. In particular we have not entered the discussion of the topology of the Picard variety of a compact Riemann surface.

In any case these notes may give some useful comments to Gunning's book and in any case our subsequent presentation at least gives some nice glimpses of the classical theory of Riemann Surfaces.

Perhaps the results are not as important as the methods. We are going to use sheaves and their cohomology groups and therefore we have included a preliminary chapter which contains what we need about sheaves.

Otherwise the material in Chapter 2 is rather self-contained. The complex analysis we are using is always standard, using such facts as the residue theorem or the Frechet topology on $\mathcal{O}(U)$ = the holomorphic functions on an open set U in the complex plane C^1 .

The differential geometry we use is also standard and since we work in two real dimensions only, the versions of Stoke's Theorem are easy and can be understood intuitively.

Let us remark that a compact Riemann Surface X is a connected and orientable compact 2-dimensional manifold. The topological structure of X is therefore wellknown, X is a sphere with a finite number of handles

CHAPTER 1
Sheaves and their Cech Cohomology

Let X be a topological space. A sheaf (of abelian groups) over X is a topological space $\mathcal{S}$ together with a mapping $\pi: \mathcal{S} \to X$, such that

1) π is a local homeomorphism

2) For each point $x \in X$, the set $S_x = \pi^{-1}(x)$ has the structure of an abelian group

3) The group operations are continuous in the topology of $\mathcal{S}$

Remark Hence the sheaf $\mathcal{S}$ consists of the sets $\{ S_x \}_{x\in X}$ and the abelian groups S_x are called the stalks of the sheaf $\mathcal{S}$. Let us illuminate condition 3) above. First, let α and β be two elements in a stalk S_x. Since S_x is an abelian group we can consider the sum $\alpha + \beta$, or the difference $\alpha - \beta$ in the abelian group S_x. At the same time 1) shows that α has a neighborhood W in $\mathcal{S}$ and β has a neighborhood W' such that $\pi: W \to U$ and $\pi: W' \to U'$ are 1-1-maps into open sets U and U' in X. Here both U and U' are neighborhoods of the point $x = \pi(\alpha) = \pi(\beta)$ and without loss of generality we can assume that $U = U'$.

If $y \in U$ we get a unique point $\alpha(y) \in \pi^{-1}(y) \cap W$ and similarly we get a unique point $\beta(y) \in \pi^{-1}(y) \cap W'$.

Here both $\alpha(y)$ and $\beta(y)$ belong to the abelian group S_y. So we can consider $\alpha(y) - \beta(y) = \sigma(y)$ say. Then 3) should mean that

$\lim_{y \to x} \sigma(y) = \alpha(x) - \beta(x) = \alpha - \beta$. More precisely, if 1) is applied to the element $\alpha - \beta$ in S_x, we get another neighborhood W" of $\alpha - \beta$ such that $\pi: W'' \to U''$ where U" is another open neighborhood of x and to each point $y \in U''$ we get a unique point $\varphi(y) \in \pi^{-1}(y) \cap W''$ and 3) means that

$\varphi(y) = \alpha(y) - \beta(y)$ if $y \in U \cap U''$ and both U and U" are sufficient--lu small open neighborhoods of x.

The local homeomorphism $\pi: \mathcal{S} \to X$ is called the projection .

1. Sections with values in a sheaf

Let $\mathcal{S}$ be a sheaf on the topological space X. If U is an open set in X we consider the family of continuous maps $f: U \to \mathcal{S}$ satisfying $2 \circ f(x) = x$ for all $x \in U$. This means that $f(x) \in S_x$ for all $x \in U$.

This family of mappings is denoted by $\Gamma(U, \mathcal{S})$ and the elements in $\Gamma(U, \mathcal{S})$ are called sections over U with values in the sheaf $\mathcal{S}$.

Let us observe the following

Lemma 1.2. Let f and $g \in \Gamma(U, \mathcal{S})$ and suppose that $f(x) = g(x)$ for some point $x \in U$. Then $f = g$ in a neighborhood of x

Proof 1) implies that the point $\alpha = f(x) = g(x)$ has a neighborhood W such that $\pi: W \to U'$ is a 1-1 map , where U' is an open neighborhood of x. We can assume that $U' \subset U$ (shrinking W if necessary). Since both f and g are continuous maps from U into $\mathcal{S}$, it follows that there exists a neighborhood U" (with $U'' \subset U'$) such that f(U") and g(U") both are contained in the neighborhood W of α.

By assumption $\pi \circ f$ and $\pi \circ g$ give the identity maps. It follows that if $y \in U''$ then $f(y) = g(y) =$ the unique point in $\pi^{-1}(y) \cap W$.

Summing up, Lemma 1.2. gives a local uniqueness principle for sections, i.e. two sections which are equal at a given point x are equal in some neighborhood of this point.

Condition 3) implies that $\Gamma(U, \mathcal{S})$ are abelian groups. In fact, let f and $g \in \Gamma(U, \mathcal{S})$. To each point $x \in U$ we get the two elements f(x) and g(x) in the abelian group S_x . Hence the sum $f(x) + g(x)$ and the difference $f(x) - g(x)$ is defined in S_x . Now 3) means that the map

$x \to f(x) - g(x)$ is a continuos mapping from U into $\mathcal{S}$ and this defines the section f - g in $\Gamma(U, \mathcal{S})$,and so on.

Let us also observe that the stalks in the sheaf $\mathcal{S}$ are recaptured from locally defined sections.

<u>Lemma</u> 1.3. Let $\alpha \in S_x$ be given, where x is some point in X. Then x has some neighborhood U and there exists some $f \in \Gamma(U, \mathcal{S})$ such that $f(x) = \alpha$

<u>Proof</u> 1) means that α has a neighborhood W such that $\pi: W \to U$ is a 1-1 and bicontinuous map from W onto an open set U in X.

If $y \in U$ is given we let f(y) be the unique point in $W \cap \pi^{-1}(y)$ and then $f: U \to \mathcal{S}$ gives a continuous map, i.e. $f \in \Gamma(U, \mathcal{S})$ and we see that $f(x) = \alpha$.

Summing up, each point $s \in \mathcal{S}$ is contained in the image of some section, and the images of all such sections form a basis for the open neighborhoods of s in the topological space $\mathcal{S}$.

1.4. <u>The zero section</u> In each abelian group S_x we have the unique zero element denoted by O_x . It is easily seen that 1) - 3) imply that the map which sedns $x \in X$ into O_x is a globally defined section and this is called the zero-section of the sheaf $\mathcal{S}$.

1.5. <u>Sheaf homomorphisms</u> Consider two sheaves $\mathcal{S}$ and $\mathcal{R}$ over X. A sheaf homomorphism $\varphi: \mathcal{S} \to \mathcal{R}$ is a continuous map from $\mathcal{S}$ into $\mathcal{R}$ satisfying: φ commutes with the projection π, i.e. $\varphi(S_x) \subset F_x$ for all x and finally, the restriction of φ to a single stalk S_x gives a group homomorphism from S_x into the abelian group F_x .

Let $\varphi: \mathcal{S} \to \mathcal{R}$ be a sheaf homomorphism and consider a section $f \in \Gamma(U, \mathcal{S})$ where U is some open set in X. Then the composition $\varphi \circ f$ is a continuous map from U into $\mathcal{R}$ and since both f and φ commute with π, it follows that $\varphi \circ f$ commutes with π. Hence

$\varphi \circ f \in \Gamma(U, \mathcal{R})$. Finally, since φ restricts to a group homomorphism from the stalks of $\mathcal{S}$ into the stalks of $\mathcal{R}$, it follows that the resultning map $\varphi^*: \Gamma(U, \mathcal{S}) \to \Gamma(U, \mathcal{R})$ is a group homomorphism.

φ^* is called the <u>induced homomorphism</u> of the groups of sections with values in $\mathcal{S}$ and $\mathcal{R}$ respectively.

1.6. <u>Subsheaves and factor sheaves</u>

Let $\varphi: \mathcal{S} \to \mathcal{R}$ be a sheaf homomorphism. The zero section in $\mathcal{R}$ is an open subset of $\mathcal{R}$ and hence the inverse image $\varphi^{-1}(0_{\mathcal{R}}) = \{ \alpha \in \mathcal{S} : \varphi(\alpha) = 0_{\pi(\alpha)} \}$ is an open subset of $\mathcal{S}$.

If we restrict the attention to a stalk then

$\varphi^{-1}(0_{\mathcal{R}}) \cap S_x = \{ \alpha \in S_x : \varphi(\alpha) = 0_x \}$ = the kernel of the group homomorphism which φ determines from S_x into F_x .

So if $\mathcal{J} = \varphi^{-1}(0_{\mathcal{R}})$, then the sets $J_x = \mathcal{J} \cap \pi^{-1}(x)$ are subgroups of $\mathcal{S}_x$ for all $x \in X$. In addition $\mathcal{J}$ is an open subset of $\mathcal{S}$ and 1) - 3) are conditions which give similar conditions on $\mathcal{J}$. We conclude that $\mathcal{J}$ is a new sheaf on X which is called the kernel of φ and denoted by $\ker(\varphi)$.

In the same way we can prove that the image of φ is a sheaf. In fact, verify first that $\mathrm{Im}(\varphi)$ is an open subset of $\mathcal{R}$, and so on.

Finally we can construct factor shaves, such as $\mathcal{S}/\mathrm{Ker}(\varphi)$. Here is the procedure for this construction:

In general, let $\mathcal{S}$ be a sheaf and let $\mathcal{J}$ be a subsheaf of $\mathcal{S}$, which means that $\mathcal{J}$ is an open subset of $\mathcal{S}$ and that the stalks J_x are subgroups of S_x for all points $x \in X$.

The <u>factor sheaf</u> (or the <u>quotient</u> sheaf) $\mathcal{S}/\mathcal{J}$ is constructed as follows: To each point $x \in X$ we get the abelian group $R_x = S_x/J_x$

and it remains to define a topology on the disjoint union $\dot{\bigcup}_{x\varepsilon X} R_x = \mathcal{R}$

Of course, the canonical map $\varphi: \mathcal{S} \to \mathcal{R}$ which to each $\alpha \varepsilon S_x$ associates its coset in $R_x = S_x/J_x$ commutes with the projection π from $\mathcal{S}$ respectivley from $\mathcal{R}$.

On $\mathcal{R}$ we introduce the quotient topology, i.e. a set W' in $\mathcal{R}$ is open if and only if $\varphi^{-1}(W')$ is open in $\mathcal{S}$. Then it is easily seen that $\mathcal{R}$ is a sheaf and that $\varphi: \mathcal{S} \to \mathcal{R}$ is a sheaf homomorphism.

Returning to the sheaf homomorphism $\varphi: \mathcal{S} \to \mathcal{R}$ we see that Im(φ) is isomorphic to the factor sheaf $\mathcal{S}$/Ker(φ). Of course, by an isomorphism of two sheaves we mean the following:

Two sheaves $\mathcal{S}$ and $\mathcal{S}'$ are isomorphic if there exists a bicontinuous map $\varphi: \mathcal{S} \to \mathcal{S}'$ which is a group isomorphism between the stalks S_x and S'_x for all $x \varepsilon X$.

1.7. Exact sequences of sheaves The diagram

$\mathcal{J} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\sigma} \mathcal{R}$ will be called an exact sequence of sheaves if the image of φ is precisely the kernel of σ.

Similarly, a longer string of sheaves will be called an exact sequence if for any two consecutive homomorphisms, the image of the first is precisely the kernel of the second.

In particular, if 0 denotes the trivial sheaf whose stalk at every point in X is the zero group, then a sequence

$0 \to \mathcal{J} \xrightarrow{\varphi} \mathcal{S} \xrightarrow{\sigma} \mathcal{R} \to 0$ is exact, if and only if φ is injective and σ is surjective and Ker(σ) = Im(φ).

An example If $\mathcal{J}$ is a subsheaf of $\mathcal{S}$, the inclusion mapping $i: \mathcal{J} \to \mathcal{S}$ is injective and the map from $\mathcal{S}$ into $\mathcal{S}/\mathcal{J}$ is surjective and

$0 \to \mathcal{J} \to \mathcal{S} \to \mathcal{S}/\mathcal{J} \to 0$ is an exact sequence of sheaves.

2. Cohomology of sheaves

Consider an open covering $\mathcal{U} = \{ U_\alpha \}$ of the space X. That is, each U_α is an open subset of X and their union $\bigcup U_\alpha = X$. No further assumptions are made, in particular some U_α's can overlap, and so on.

To each finite set of indices, say $\alpha_o \dots \alpha_p$ we can consider the (possibly empty) intersection $U_{\alpha_o} \cap \dots \cap U_{\alpha_p}$. When this set is non-empty it is denoted by $U_{\alpha_o \dots \alpha_p}$ and following the terminology used in algebraic topology, the set $U_{\alpha_o \dots \alpha_p}$ is called the support of a p-simplex determined by the open covering $\mathcal{U}$.

Let us now consider a sheaf $\mathcal{S}$ on X. A p-cochain of $\mathcal{U}$ with values in $\mathcal{S}$ is a function f which associates to every p-simplex $\{ U_{\alpha_o}, \dots U_{\alpha_p} \}$ a section $f_{\alpha_o \dots \alpha_p} \in \Gamma(U_{\alpha_o \dots \alpha_p}, \mathcal{S})$

The set of all such p-cochains is denoted by $C^p(\mathcal{U}, \mathcal{S})$.

Observe that $C^p(\mathcal{U}, \mathcal{S})$ is an abelian group. In fact, by definition $C^p(\mathcal{U}, \mathcal{S})$ is the direct sum of the abelain groups $\{ \Gamma(U_{\alpha_o \dots \alpha_p}, \mathcal{S})$.

2.1. The coboundary operator δ

To each $p \geqslant 0$ we can define a mapping $\delta: C^p(\mathcal{U}, \mathcal{S}) \to C^{p+1}(\mathcal{U}, \mathcal{S})$ which is called the coboundary operator. Namely, let $f \in C^p(\mathcal{U}, \mathcal{S})$ be given and consider a (p+1)-simplex $\{ U_{\alpha_o \dots \alpha_{p+1}} \}$. Then we can define a section $(\sigma f)_{\alpha_o \dots \alpha_{p+1}} \in \Gamma(U_{\alpha_o \dots \alpha_{p+1}}, \mathcal{S})$ as follows:

$$(\delta f)_{\alpha_o \dots \alpha_{p+1}} = \sum_{i=0}^{i=p} (-1)^i (\rho f)_{\alpha_o \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_{p+1}}$$

where (ρf) denotes the restriction of the section

$f_{\alpha_o \dots a_{i-1}, \alpha_{i+1} \dots \alpha_{p+1}} \in \Gamma(U_{\alpha_o \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_{p+1}})$ to the smaller open set $U_{\alpha_0 \dots \alpha_{p+1}}$ = the intersection of U_{α_i} and $U_{\alpha_0 \dots \alpha_{i-1}, \alpha_{i+1} \dots \alpha_{p+1}}$ for each $0 \leqslant i \leqslant p+1$.

The choice of the signs $(-1)^i$ imply easily that $\delta^2 = 0$.

2.2. Cocycles and coboundaries

To each $p \geqslant 0$ we have the coboundary operator $\delta: C^p(\mathcal{U},\mathcal{S}) \to C^{p+1}(\mathcal{U},\mathcal{S})$ and its definition shows that δ is a group homomorphism. Let

$Z^p(\mathcal{U},\mathcal{S})$ be the kernel of δ and $B^{p+1}(\mathcal{U},\mathcal{S}) = \delta(C^p(\mathcal{U},\mathcal{S}))$ the image of δ.

The groups $Z^p(\mathcal{U},\mathcal{S})$ are called p-cocycles and the groups $B^p(\mathcal{U},\mathcal{S})$ are called p-coboundaries.

Since $\delta^2 = 0$ we get the inclusions $B^p(\mathcal{U},\mathcal{S}) \subset Z^p(\mathcal{U},\mathcal{S})$ for all $p \geqslant 0$. The quotient groups

$Z^p(\mathcal{U},\mathcal{S})/B^p(\mathcal{U},\mathcal{S})$ are denoted by $H^p(\mathcal{U},\mathcal{S})$ and they are called the cohomology groups of $\mathcal{U}$ with coefficients in the sheaf $\mathcal{S}$.

We begin to study these cohomology groups $H^p(\mathcal{U},\mathcal{S})$. In particular we study their behavious as we pass from one open covering $\mathcal{U}$ to another open covering $\mathcal{V}$ on the space X. We begin with

Lemma 2.3. $H^o(\mathcal{U},\mathcal{S}) \simeq \Gamma(X,\mathcal{S})$

Proof By definition $H^o(\mathcal{U},\mathcal{S}) = Z^o(\mathcal{U},\mathcal{S})$ beauce we have no coboundarie-s in $C^o(\mathcal{U},\mathcal{S})$. A zero-cohain f is a function which to each open set U_α associates a section $f_\alpha \in \Gamma(U_\alpha,\mathcal{S})$ and the condition that $\delta f = 0$ means that $f_\alpha = f_\beta$ when $U_\alpha \cap U_\beta$ is non-empty. It follows that these locally defined sections $\{ f_\alpha \}$ build up a globally defined section (F) over X where the restriction of F to each open set U_α is f_α.

This defines the map from $f = \{ f_\alpha \} \in Z^0(\mathcal{U},\mathcal{S})$ into $\Gamma(X,\mathcal{S})$.

The map is obviously injective, because $F = 0$ in $\Gamma(X,\mathcal{S})$ means that each $f_\alpha = 0$ in $\Gamma(U_\alpha,\mathcal{S})$. Finally, if we start from $F \in \Gamma(X,\mathcal{S})$ we consider the restrictions $f_\alpha = F|_{U_\alpha}$ and then $f = \{ f_\alpha \} \in Z^0(\mathcal{U},\mathcal{S})$. Hence we have proved Lemma 2.3.

In order to have a cohomology theory associated intrinsically to the space X, it is necessary to consider various possible coverings of X. Let us begin with

<u>Definition</u> 2.4. An open covering $\mathcal{V} = \{ V_\alpha \}$ is called <u>a refinement</u> of the covering $\mathcal{U} = \{ U_\beta \}$, if there exists a mapping $\sigma : \mathcal{V} \to \mathcal{U}$ which to each α gives some $\sigma(\alpha)$ such that the open set $V_\alpha \subset U_{\sigma(\alpha)}$. The index mapping σ is called the refining mapping.

Of course, $\mathcal{V}$ can be a refinement of $\mathcal{U}$ by various different refining mappings and this will be studied in Lemma 2.5. below.

Notice first that if $\sigma : \mathcal{V} \to \mathcal{U}$ is a refining mapping, then σ induces a mapping from $C^p(\mathcal{U},\mathcal{S})$ into $C^p(\mathcal{V},\mathcal{S})$ for all $p \geq 0$ as follows:

If $f \in C^p(\mathcal{U},\mathcal{S})$ is given and if we consider some open set $V_{\alpha_0 \ldots \alpha_p}$ then we get a section $(\sigma f)_{\alpha_0 \ldots \alpha_p} \in \Gamma(V_{\alpha_0 \ldots \alpha_p})$ as follows:

To each $0 \leq i \leq p$ we get $V_{\alpha_i} \subset U_{\sigma(\alpha_i)}$ and in the p-cochain f we have the section $f_{\sigma(\alpha_0)\ldots\sigma(\alpha_p)} \in \Gamma(U_{\sigma(\alpha_0)\ldots\sigma(\alpha_p)}, \mathcal{S})$. Now $V_{\alpha_0 \ldots \alpha_p} \subset U_{\sigma(\alpha_0)\ldots\sigma(\alpha_p)}$ and hence we can restrict this section to $V_{\alpha_0 \ldots \alpha_p}$, i.e. put $(\sigma f)_{\alpha_0 \ldots \alpha_p} = f_{\sigma(\alpha_0)\ldots\sigma(\alpha_p)}$ restricted to $V_{\alpha_0 \ldots \alpha_p}$.

The resulting map from $C^p(\mathcal{U},\mathcal{S})$ into $C^p(\mathcal{V},\mathcal{S})$ is obviously a group homomorphism which commutes with the coboundary operators δ on $C^*(\mathcal{U}, \mathcal{S})$ and $C^*(\mathcal{U}, \mathcal{S}')$.

In particular $\sigma(Z^p(\mathcal{U}, \mathcal{S}')) \subset Z^p(\mathcal{V}, \mathcal{S})$ and similarly for the coboundaries. We conclude that there exist homomorphisms

$\sigma^*: H^p(\mathcal{U}, \mathcal{S}) \to H^p(\mathcal{V}, \mathcal{S})$ for all $p > 0$.

Summing up, a refining map $\sigma: \mathcal{V} \to \mathcal{U}$ induces group homomorphisms from $H^p(\mathcal{U}, \mathcal{S})$ into $H^p(\mathcal{V}, \mathcal{S}')$ for all $p \geq 0$.

Now we can prove that these induced maps on the cohomology groups are independent of the particular refining map.

<u>Lemma</u> 2.5. Let σ and η be two refining maps from $\mathcal{V}$ into $\mathcal{U}$. Then their induced maps from $H^p(\mathcal{U}, \mathcal{S}') \to H^p(\mathcal{V}, \mathcal{S}')$ are equal for all $p \geq 0$.

<u>Proof</u> If $p = 0$, then Lemma 2.3. shows that the induced maps both correspond to the identity maps on $\Gamma(X, \mathcal{S}')$.

It remains to study the case $p > 0$. In this case we can construct a map $\theta : C^p(\mathcal{U}, \mathcal{S}') \to C^{p-1}(\mathcal{V}, \mathcal{S}')$ as follows:

Let $f \in C^p(\mathcal{U}, \mathcal{S}')$ be given and consider some open set $V_{\alpha_0 \dots \alpha_{p-1}}$ and define

$$(\theta f)_{\alpha_0 \dots \alpha_{p-1}} = \sum_{j=0}^{j=p-1} (-1)^j \, (\rho f)_{\sigma(\alpha_0) \dots \sigma(\alpha_j), \eta(\alpha_j) \dots \eta(\alpha_{p-1})}$$

That is, to each $0 \leq j \leq p-1$ we can consider the support of the p-simplex $(U_{\sigma(\alpha_0) \dots \sigma(\alpha_j), \eta(\alpha_j) \dots \eta(\alpha_{p-1})})$ and restrict sections on this support to the subset $V_{\alpha_0} \dots V_{\alpha_p}$ of $U_{\sigma(\alpha_0) \dots \sigma(\alpha_j) \eta(\alpha_j) \dots \eta(\alpha_{p-1})}$. The point is simply that V_{α_j} now is a subset of the intersection

$U_{\sigma(\alpha_j)} \cap U_{\eta(\alpha_j)}$ for all $0 \leq j \leq p-1$.

Using the coboundary operators we see that

$\sigma \circ \theta$ is a map from $C^p(\mathcal{U}, \mathcal{S})$ into $C^p(\mathcal{V}, \mathcal{S})$ and similarly

$\Theta \circ \delta$ maps $C^p(\mathcal{U},\mathcal{S})$ into $C^p(\mathcal{V},\mathcal{S})$.

We leave out the straightforward comutation which shows

<u>Sublemma</u> We have: $\delta \circ \Theta = \eta^* - \sigma^* - \Theta \circ \delta$ for all $p \geqslant 1$.

In particular, let $f \in Z^p(\mathcal{U},\mathcal{S})$ so that $\delta f = 0$. It follows that $\eta^*(f) = \sigma^*(f) + \delta(\Theta(f))$ in $C^p(\mathcal{V},\mathcal{S})$.

Here $\delta(\Theta(g)) \in B^p(\mathcal{U},\mathcal{V})$ and we conclude that $\eta^*(f)$ and $\sigma^*(f)$ have the same images in $H^p(\mathcal{U},\mathcal{V})$.

Summing up, Lemma 2.5. shows that if $\mathcal{V} < \mathcal{U}$, then there exists a unique homomorphism from $H^p(\mathcal{U},\mathcal{S})$ into $H^p(\mathcal{V},\mathcal{S})$ for each $p \geqslant 0$, which can be defined by some refining map $\sigma: \mathcal{V} \to \mathcal{U}$.

The set of all coverings of X is partially ordered by the relation $<$ and we can introduce the <u>direct limit groups</u>

$$H^p(X,\mathcal{S}) = \mathrm{dir.lim}_{\mathcal{U}}\, H^p(\mathcal{U},\mathcal{S}) \quad \text{for each } p \geqslant 0.$$

<u>Definition</u> 2.6. $H^p(X,\mathcal{S})$ is an abelian group which is called the <u>p-th cohomology group</u> of the space X with values in the sheaf $\mathcal{S}$.

3. <u>Fine sheaves</u>

In Section 2 we have defined the cohomology groups $H^p(X,\mathcal{S})$ of a sheaf $\mathcal{S}$. Here we try to compute these groups in another way.

From now on we make a restrictive assumption on the topological space X. Namely, we assume that X is <u>paracompact</u> and Hausdorff. This means that every open covering of X has a refinement which is a locally finite covering and it is sufficient to consider locally finite coverings when we pass to the direct limit which defines $H^p(X,\mathcal{S})$

Let us first observe that if $\eta : \mathcal{S} \to \mathcal{R}$ is a sheaf homomorphism then η determines maps from $C^p(\mathfrak{U}, \mathcal{S}) = \oplus \Gamma(U_{\alpha_0 \ldots \alpha_p}, \mathcal{S})$ into $C^p(\mathfrak{U}, \mathcal{R})$ for every open covering $\mathfrak{U}$ of X. It is obvious that this homomorphism commutes with the coboundary operator and we conclude that η induces a homomorphism $\eta^* : H^p(\mathfrak{U}, \mathcal{S})$ into $H^p(\mathfrak{U}, \mathcal{R})$ for all $p \geqslant 0$. Passing to the direct images we get the induced mappings

$$\eta^* : H^p(X, \mathcal{S}) \longrightarrow H^p(X, \mathcal{R}) \quad \text{for all } p .$$

Let us now consider an exact sequence of sheaves:

$0 \to \mathcal{S}_1 \to \mathcal{S}_2 \to \mathcal{S}_3 \to 0$. This exact sequence gives rise to a long exact sequence of abelian groups :

$$0 \to H^0(X, \mathcal{S}_1) \to H^0(X, \mathcal{S}_2) \to H^0(X, \mathcal{S}_3) \to H^1(X, \mathcal{S}_1) \to H^1(X, \mathcal{S}_2) \to \ldots$$

This will be proved later on, using the famous Exact sequence in Homology which we announce here :

In general, let $A^* : \quad 0 \to A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \to \ldots$ be a complex of abelian groups, i.e. each A_v is an abelian group and $d: A_v \to A_{v+1}$ are group homomorphisms and $d^2 = 0$. Now we can consider an exact sequence of complexes : $0 \to A^* \to B^* \to C^* \to 0$. This means that we have a Commutative diagram

$$\begin{array}{ccccccc}
\ldots & C^{v-1} & \to & C^v & \to & C^{v+1} & \to \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & B^{v-1} & \to & B^v & \to & B^{v+1} & \to \\
 & \uparrow & & \uparrow & & \uparrow & \\
 & A^{v-1} & \to & A^v & \to & A^{v+1} & \to
\end{array}$$

where the rows are complexes and each column is an exact sequence $0 \to A^v \to B^v \to C^v \to 0$ of abelian groups.

Introduce the cohomology groups $H^v(A^*) = \mathrm{Ker}(d)\, A^v / d(A^{v-1})$ and similarly for the two complexex B^* and C^*. Then we get a long exact sequence

$$\ldots \to H^{v-1}(C^*) \to H^v(A^*) \to H^v(B^*) \to H^v(C^*) \to H^{v+1}(A^*) \to \ldots.$$

Let us now introduce a family of sheaves whose cohomology vanishes when $p \geqslant 0$. We begin with

<u>Definition</u> 3.1. Let $\mathcal{U} = \{ U_\alpha \}$ be a locally finite open covering of X. A partition of the unity of a sheaf $\mathcal{S}$, subordinate to the covering is a family of sheaf homomorphisms : $\eta_\alpha : \mathcal{F} \to \mathcal{F}$ such that

1) $\eta_\alpha(F_x) = 0$ for all $x \in X \setminus U_\alpha$ and

2) $\sum_\alpha \eta_\alpha(s) = s$ for every $s \in \mathcal{F}$.

<u>Definition</u> 3.2. A sheaf $\mathcal{F}$ for which there exist partitions of the unity, subordinate for every locally finite covering of X, <u>is called a fine sheaf</u>

With these notations we can prove

<u>Lemma</u> 3.3. If $\mathcal{F}$ is a fine sheaf then $H^p(X, \mathcal{F}) = 0$ for each $p \geq 1$.

<u>Proof</u> Let $\mathcal{U}$ be a locally finite covering of X. It is sufficient to prove that $H^p(\mathcal{U}, \mathcal{F}) = 0$ for some given $p > 0$. To prove this we consider a p-cocycle $f \in Z^p(\mathcal{U}, \mathcal{F})$ and when α is fixed and $\{U_{\beta_o \ldots \beta_{p-1}}\}$ is a (p-1)-simplex we can consider the section

$$f_{\alpha, \beta_o \ldots \beta_{p-1}} \quad \text{which exist on } U_{\beta_o \ldots \beta_{p-1}} \cap U_\alpha .$$

~~have restrictions to the smaller set~~ $\cancel{U_\alpha U_{\beta_o \ldots \beta_{p-1}}}$ Since $\eta_\alpha(\ _x) = 0$ when $x \in X \setminus U_\alpha$ we see that the section $\eta_\alpha(f_{\alpha, \beta_o \ldots \beta_{p-1}})$ can be extended to a section over $U_{\beta_o \ldots \beta_{p-1}}$, where it is the zero -section in the complementary set $U_{\beta_o \ldots \beta_{p-1}} \setminus U_{\alpha, \beta_o \ldots \beta_{p-1}}$

In this way the p-cochain f defines the (p-1)-cochain

$F_\alpha = \{ \eta_\alpha(f_{\alpha,\beta_0\ldots\beta_{p-1}}) = F_{\beta_0\ldots\beta_{p-1}} \}$.

If η_α^* is the induced map on $H^p(\mathcal{U}, \mathcal{F})$ which arises from the sheaf homomorphism $\eta_\alpha : \mathcal{F} \to \mathcal{F}$, then it is an easy computation to prove that $\delta F_\alpha = \eta_\alpha^* f$ for all $f \in Z^p(\mathcal{U}, \mathcal{F})$

Finally, since the covering $\mathcal{U}$ is locally finite, it follows that $\Sigma_\alpha F_\alpha \in C^{p-1}(\mathcal{U}, \mathcal{F})$ and $\delta(\Sigma_\alpha F_\alpha) = \Sigma_\alpha \eta^*_\alpha f = f$, where the last equality follows from 2) in Definition 3.1. above.

Hence $f \in B^p(\mathcal{U}, \mathcal{F})$ and we have proved that the quotient group $H^p(\mathcal{U}, \mathcal{F}) = Z^p/B^p = 0$, as required.

Let us now apply Lemma 3.3. to compute the cohomology of a sheaf $\mathcal{S}$. The idea is to construct <u>a fine resolution</u> of the sheaf $\mathcal{S}$.

That is, we construct an exact sequence of sheaves

$0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \ldots$, where $\mathcal{S}^v$ are fine sheaves for all $v \geq 0$. Let us begin with

3.4. <u>The existence of fine resolutions</u>

Let $\mathcal{S}$ be a sheaf on X. If U is an open set in X we can consider the family $\Gamma_d(U, \mathcal{S})$ of <u>all functions</u> $f: U \to \mathcal{S}$ satisfying $f(x) = x$ for $x \in U$.

That is, we do not impose any continuity condition. $\Gamma_d(U, \mathcal{S})$ are abelian groups for all open sets and we get abelian groups S^o_x for each point $x \in X$ when we take the direct limits

$S^o_x = \mathrm{di.rlim}_U \Gamma_d(U, \mathcal{S})$ where U moves over all neighborhoods to x.

The abelian groups $\{ S^o_x \}_{x \in X}$ are the stalks in a sheaf $\mathcal{S}^o$ whose sections $\Gamma(U, \mathcal{S}^o) = \Gamma_d(U, \mathcal{S})$ for every open set U.

$\mathcal{S}^{o}$ is called the sheaf associated to the discontinuous sections of $\mathcal{S}$.

It is obvious that $\mathcal{S}^{o}$ is a fine sheaf. Also, since $\Gamma(U, \mathcal{S})$ are subgroups of $\Gamma(U, \mathcal{S}^{o})$ and each stalk $S_x = \text{dir.lim}_U\, \Gamma(U, \mathcal{S})$ (See Lemma 1.2. and 1.3.) it follows that S_x are subgroups of $S^{o}_{x} = \text{dir.lim}_U\, \Gamma_d(U, \mathcal{S})$ for every $x \in X$.

Hence $\mathcal{S}$ is a subsheaf of $\mathcal{S}^{o}$. Now we can consider the quotient sheaf $\mathcal{S}^{o}/\mathcal{S}$ and repeat the construction to get $(\mathcal{S}^{o}/\mathcal{S})^{o} = \mathcal{S}^{1}$ = a fine sheaf and the exact sequence $0 \to \mathcal{S} \to \mathcal{S}^{o} \to \mathcal{S}^{1}$ and continue. Summing up, we have proved

<u>Lemma</u> 3.4. If $\mathcal{S}$ is a sheaf then there exists a fine resolution $0 \to \mathcal{S} \to \mathcal{S}^{o} \to \mathcal{S}^{1} \to \ldots$ where $\{\mathcal{S}^{o}, \mathcal{S}^{1}, \ldots\}$ are fine sheaves.

~~4. The long exact sequence of cohomology~~

Consider an exact sequence of sheaves $0 \quad \varphi \quad \sigma \quad 0$ and let be an open covering of X. To each $p > 0$ we get the abelian groups of p-cochains $C^p(\ ,\ )$, $C^p(\ ,\ )$ and $C^p(\ ,\ )$ and the induced group homomorphisms

$\varphi : C^p(\ ,\ ) \quad C^p(\ ,\ )$ and $\sigma : C^p(\ ,\ ) \quad C^p(\ ,\ )$

Here $\sigma \circ \varphi = 0$ and we also get the induced homomorphsims $\varphi^{*} : H^p(\ ,\ ) \quad H^p(\ ,\ )$ and $\sigma^{*} : H^p(\ ,\ ) \quad H^p(\ ,\ )$ as described in the beginning of section 3.

It turns out that we can construct homomorphisms from

$H^p(\ ,\ ) \quad H^{p+1}(\ ,\ )$ for all $p > 0$ and in this way get a long exact sequence

$$0 \quad H^{o}(\ ,\ ) \quad H^{o}(\ ,\ ) \quad H^{o}(\ ,\ ) \quad H^{1}(\ ,\ ) \quad \ldots$$

4. The long exact sequence of cohomology and Leray's Theorem

If U is an open subset of X and $\mathcal{S}$ is a sheaf on X, then $\mathcal{S}$ can be restricted to a sheaf on U. In fact, put $S_{|U} = \pi^{-1}(U) = \bigcup_{x \in U} S_x$. Obviously $S_{|U}$ is a sheaf on U and we can compute its cohomology groups which we denote by $H^p(U,\mathcal{S})$.

In order to avoid too many notations and rather tedious passages to direct limits we shall restrict the attention to sheaves which have a locally trivial cohomology, as explained below. Let us also remark that this family of sheaves actually plays an important role, and Leray's Theorem which is proved later on plays a crucial role when we begin to study certain sheaves on compact Riemann surfaces.

4.1. Leray coverings of sheaves

Let $\mathcal{U} = \{ U_\alpha \}$ be an open and locally finite covering of X and let $\mathcal{S}$ be a sheaf on X. Then $\mathcal{U}$ is called a Leray covering with respect to the sheaf $\mathcal{S}$ if the following is true : $H^p(U_{\alpha_0 \dots \alpha_q}, \mathcal{S}) = 0$ for every $p \geqslant 1$ and every non-empty intersection of finitely many sets $U_{\alpha_0}, \dots U_{\alpha_q}$ from the covering $\mathcal{U}$.

Definition 4.2. A sheaf $\mathcal{S}$ satisfies Leray's condition if every open covering of X has a refinement $\mathcal{U}$ which is a Leray covering with respect to the sheaf $\mathcal{S}$.

Remark Recall that we assume that X is paracompact. So in the definitions above we have restricted the attention to locally finite coverings of X.

With these notations we can announce

Leray's Theorem Let $\mathcal{S}$ be a sheaf on X and let $\mathcal{U}$ be a Leray covering of $\mathcal{S}$. Then $H^p(X,\mathcal{S}) \cong H^p(\mathcal{U},\mathcal{S})$ for all p .

Before we enter the proof we present the details of the following result:

Proposition 4.3. Let $0 \xrightarrow{\varphi} \mathcal{S} \to \mathcal{F} \xrightarrow{\sigma} \mathcal{R} \to 0$ be an exact sequence of sheaves and assume that $H^1(X, \mathcal{S}) = 0$. Then

$0 \to \Gamma(X, \mathcal{S}) \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{R}) \to 0$ is an exact sequence of abelian groups.

Proof $\mathcal{S}$ can be identified with a subsheaf of $\mathcal{F}$ and the inclusion $\Gamma(X, \mathcal{S}) \to \Gamma(X, \mathcal{F})$ follows. Suppose now that $f \in \Gamma(X, \mathcal{F})$ and that $\sigma f = 0$ in $\Gamma(X, \mathcal{R})$. At a single stalk this means that $\sigma(f(x)) = 0$ in R_x and since $0 \to S_x \to F_x \to R_x \to 0$ is exact, it follows that $f(x) \in S_x$ for all $x \in X$. This means that $f \in \Gamma(X, \mathcal{S})$ and we have proved that $\mathrm{Ker}(\varphi) = \sigma(\Gamma(X, \mathcal{S}))$. It remains only to prove that σ maps $\Gamma(X, \mathcal{F})$ onto $\Gamma(X, \mathcal{R})$.

So let $g \in \Gamma(X, \mathcal{R})$ be given. If $x \in X$ then $g(x) \in R_x$ and we can find $\xi_x \in F_x$ such that $\sigma(\alpha_x) = g(x)$.

Lemma 1.3. gives a neighborhood U^x of x and some section $f^x \in \Gamma(U^x, \mathcal{F})$ such that $f^x(x) = \xi_x$ in F_x.

Now $\sigma(f^x) \in \Gamma(U^x, \mathcal{R})$ and this section is equal to g at the point x. Lemma 1.2. shows that they are equal in a neighborhood of x.

Summing up, we have proved that to each point x there exists some neighborhood U^x of x and some $f \in \Gamma(U^x, \mathcal{F})$ such that $\sigma(f^x) = g$ in U^x.

These open sets cover X and passing to a locally finite refinement we can assume that there exists a locally finite covering $\mathcal{U} = \{ U_\alpha \}$ and sections $\{ f_\alpha \in \Gamma(U_\alpha, \mathcal{F}) \}$ such that $\sigma(f_\alpha) = g$ in U_α.

Now we consider the differences $f_\alpha - f_\beta$ in $U_\alpha \cap U_\beta$. Since $\sigma(f_\alpha) = \sigma(f_\beta)$ in this intersection, it follows that

$f_\alpha - f_\beta \in \Gamma(U_{\alpha\beta}, \mathcal{S})$ and we put $h_{\alpha\beta} = f_\alpha - f_\beta$. Obviously the family $\{ h_{\alpha\beta} \}$ belongs to $Z^1(\mathcal{U}, \mathcal{S})$.

Of course we can pass to a refinement of the covering $\mathcal{U}$ and get a similar cocycle in $Z^1(\mathcal{U},\mathcal{S})$.

Since $H^1(X,\mathcal{S}) = 0$ by assumption, it follows that we can choose the covering $\mathcal{U}$ in such a way that the resulting cocycle $\{ h_{\alpha\beta} = f_\alpha - f_\beta \}$ belongs to $B^1(\mathcal{U},\mathcal{S})$.

Hence we can put $f_\alpha - f_\beta = s_\alpha - s_\beta$ where $s_\alpha \in \Gamma(U_\alpha, \mathcal{S})$ for all α.

But then the family $\{ f_\alpha - s_\alpha \}$ give a globally defined section $F \in \Gamma(X,\mathcal{F})$ (exactly as in the proof of Lemma 2.3.) and we see that $\sigma(F) = g$.

Using Proposition 4.3. and the Long exact sequence arising from an exact sequence of complexes of abelian groups, we can begin to prove

Proposition 4.4. Let $0 \xrightarrow{\varphi} \mathcal{S} \to \mathcal{F} \xrightarrow{\sigma} \mathcal{R} \to 0$ be an exact sequence of sheaves and let $\mathcal{U}$ be a Leray covering with respect to $\mathcal{S}$. Then we get the long exact sequence

$$0 \to \Gamma(X,\mathcal{S}) \to \Gamma(X,\mathcal{F}) \to \Gamma(X,\mathcal{R}) \to H^1(\mathcal{U},\mathcal{S}) \to H^1(\mathcal{U},\mathcal{F}) \to H^1(\mathcal{U},\mathcal{R}) \to$$
$$\to H^2(\mathcal{U},\mathcal{S}) \to H^2(\mathcal{U},\mathcal{F}) \to \dots$$

Proof Let us put $A^p = C^p(\mathcal{U},\mathcal{S})$, $B^p = C^p(\mathcal{U},\mathcal{F})$ and finally we let C^p = the image $\sigma(C^p(\mathcal{U},\mathcal{F}))$ in $C^p(\mathcal{U},\mathcal{R})$

We observe the following

Sublemma $C^p = C^p(\mathcal{U},\mathcal{R})$ for all p.

Proof When $p \geqslant 0$ then $C^p(\mathcal{U}, *)$ are the direct sum of $\Gamma(U_{\alpha_0 \dots \alpha_p}, *)$. Since $\mathcal{U}$ is a Leray covering with respect to $\mathcal{S}$, we know that $H^1(U_{\alpha_0 \dots \alpha_p}, \mathcal{S}) = 0$ for every p-simplex and Proposition 4.3. implies that $0 \to \Gamma(U_{\alpha_0 \dots \alpha_p},\mathcal{S}) \to \Gamma(U_{\alpha_0 \dots \alpha_p},\mathcal{F}) \to \Gamma(U_{\alpha_0 \dots \alpha_p},\mathcal{R}) \to 0$ is an exact sequence. Passing to the direct sums we conclude that

$$0 \to C^p(\mathcal{U},\mathcal{S}) \to C^p(\mathcal{U},\mathcal{F}) \to C^p(\mathcal{U},\mathcal{R}) \to 0 \quad \text{are exact}$$

sequences for all $p \geqslant 0$.

Proof continued Using the notations above and the coboundary operator δ which commutes with the maps φ and σ induce on $C^p(\mathcal{U}, \mathcal{S})$ and $C^p(\mathcal{U}, \mathcal{F})$ we get the following commutative diagram

$$\begin{array}{ccccccc}
C^0 & \to & C^1 & \to & & & \\
\uparrow & & \uparrow & & \uparrow & & \\
B^0 & \to & B^1 & \to & B^2 & \to & \cdots \\
\uparrow & & \uparrow & & & & \\
A^0 & \to & A^1 & \to & A^2 & \to & \cdots.
\end{array}$$

where the columns are exact and in each row the cohomology groups are given by $\{H^*(\mathcal{U}, *)\}$ and so on. Hence we can read off Proposition 4.4. from the Exact sequence of homology as stated in Section 3.

Corollary 4.5. Let $0 \to \mathcal{S} \to \mathcal{F} \to \mathcal{R} \to 0$ be an exact sequence of sheaves where $\mathcal{S}$ satisfies Leray's condition. Then we get the long exact sequence $0 \to H^0(X, \mathcal{S}) \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{R}) \to H^1(X, \mathcal{S}) \to \cdots$

Proof Since $\mathcal{S}$ satisfies Leray's conditions we can obtain the exact sequences above, using arbitrary fine coverings $\mathcal{U}$ of X. So when we compute the cohomology groups by direct limits we can read off Corollary 4.5. form Proposition 4.4.

Using Corollary 4.5. we can prove that the cohomology of a sheaf can be computed using fine resolutions. The result is

Lemma 4.6. Let $\mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^1 \to \cdots$ be a fine resolution of a sheaf $\mathcal{S}$ satisfying Leray's condition. Then the cohomology groups $\{H^p(X, \mathcal{S}) : p \geqslant 0\}$ are the cohomology groups in the complex

$$0 \to \Gamma(X, \mathcal{S}^0) \to \Gamma(X, \mathcal{S}^1) \to \Gamma(X, \mathcal{S}^2) \to \cdots$$

Proof Since $\{\mathcal{S}^v\}$ are fine sheaves we know that $H^p(X, \mathcal{S}^j) = 0$ for all $p \geq 1$ and we recall that $H^0(X, \mathcal{S}^j) = \Gamma(X, \mathcal{S}^j)$ for all j.

To prove Lemma 4.6. we first consider the exact sequence of sheaves

$0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^0/\mathcal{S} \to 0$ and then Corollary 4.5.

gives the exact sequence $0 \to \Gamma(X, \mathcal{S}) \to \Gamma(X, \mathcal{S}^0) \to \Gamma(X, \mathcal{S}^0/\mathcal{S}) \to H^1(X, \mathcal{S}) \to 0$

If we let $\mathcal{B}^2 = d(\mathcal{S}^1)$ be the image of the sheaf $\mathcal{S}^1$ under the map $\mathcal{S}^1 \xrightarrow{d} \mathcal{S}^2$, then $0 \to \mathcal{S}^0/\mathcal{S} \to \mathcal{S}^1 \to \mathcal{B}^2 \to 0$ is exact and again Corollary 4.5. gives the exact sequence $0 \to \Gamma(X, \mathcal{S}^0/\mathcal{S}) \to \Gamma(X, \mathcal{S}^1) \to \Gamma(X, \mathcal{B}^2) \to$ $\to H^1(X, \mathcal{S}^0/\mathcal{S}) \to 0$

The kernel of the map from $\Gamma(X, \mathcal{S}^1) \to \Gamma(X, \mathcal{S}^2)$ is denoted by Z^1.

Of course Z^1 = the kernel of the map from $\Gamma(X, \mathcal{S}^1)$ into $\Gamma(X, \mathcal{B}^2)$ since $\mathcal{B}^2 = d(\mathcal{S}^1)$ and hence the last exact sequence above gives

$Z^1 \cong \Gamma(X, \mathcal{S}^0/\mathcal{S})$ and then the first exact sequence

shows that $H^1(X, \mathcal{S}) \cong Z^1/B^1$ where B^1 = the image of $\Gamma(X, \mathcal{S}^0)$ in $\Gamma(X, \mathcal{S}^0/\mathcal{S})$ and since $\mathcal{S}^0/\mathcal{S} = d(\mathcal{S}^0)$ is a subsheaf of $\mathcal{S}^1$, this means that B^1 = the image $d(\Gamma(X, \mathcal{S}^0))$ in $\Gamma(X, \mathcal{S}^1)$.

By definition the factor grpup Z^1/B^1 is a cohomology group in the complex $0 \to \Gamma(X, \mathcal{S}^0) \to \Gamma(X, \mathcal{S}^1) \to \dots$ and hence our discussion has proved

that $H^1(X, \mathcal{S}) \cong H^1(\Gamma(X, \mathcal{S}^*))$ = the 1st cohomology group of the complex $\Gamma(X, \mathcal{S}^*)$.

The case when $p > 1$ is proved in the same way. In fact, now we can repeat the discussion above, with $\mathcal{S}$ replaced by $\mathcal{S}^0/\mathcal{S}$ and starting with the fine resolution $0 \to \mathcal{S}^0/\mathcal{S} \to \mathcal{S}^1 \to \mathcal{S}^2 \to$ and conclude that

$H^2(\Gamma(X, \mathcal{S}^*)) \cong H^1(X, \mathcal{S}^0/\mathcal{S})$.

Finally, we apply Corollary 4.5. to $0 \to \mathcal{S} \to \mathcal{S}^0 \to \mathcal{S}^0/\mathcal{S} \to 0$ and writing out the whole long exact sequence which contains

$$H^1(X,\mathcal{S}^o) \to H^1(X,\mathcal{S}^o/\mathcal{S}) \to H^2(X,\mathcal{S}) \to H^2(X,\mathcal{S}^o)$$

and using the fact that $\mathcal{S}^o$ is fine, we get

$H^1(X,\mathcal{S}^o/\mathcal{S}) \cong H^2(X,\mathcal{S})$ and hence we have proved that

$H^2(X,\mathcal{S}) \cong H^2(\Gamma(X,\mathcal{S}^*))$ and the case when $p > 2$ is now proved

in a similar way where we leave out the detials as an exercise.

4.7. <u>Remark</u> Of course, the proof above shows that if we consider an exact sequence $0 \to \mathcal{S} \to \mathcal{F}^o \to \mathcal{F}^1 \to \mathcal{F}^2 \to \ldots$ where the sheaves $\{\mathcal{F}^j\}$ satisfy $H^p(X,\mathcal{F}^j) = 0$ for all $p > 0$ and all j, then the cohomology of the complex $0 \to \Gamma(X,\mathcal{F}^o) \to \Gamma(X,\mathcal{F}^1) \to \ldots$ computes the cohomology groups $\{H^p(X,\mathcal{S})\}$

We finish this section by

<u>Proof of Leray's Theorem</u>

Let $\mathcal{U}$ be a Leray covering of the sheaf $\mathcal{S}$. Choose a fine resolution

$$0 \to \mathcal{S} \to \mathcal{S}^o \to \mathcal{S}^1 \to \mathcal{S}^2 \to \ldots$$

Since $H^p(U_{\alpha_o \ldots \alpha_q}, \mathcal{S}) = 0$ for all $p > 0$ and all finite intersections of sets from $\mathcal{U}$, we see that Lemma 4.6. and the same argument as in the proof of the Sublemma in Proposition 4.4. implies that

$0 \to C^p(\mathcal{U},\mathcal{S}) \to C^p(\mathcal{U},S^o) \to C^p(\mathcal{U},S^1) \to \ldots$ are exact

sequences for all $p \geq 0$. Now we introduce the coboundary maps and the double complex

$$\begin{array}{ccccc}
\uparrow & & \uparrow & & \\
C^2(\mathcal{U},\mathcal{S}^o) & \to & C^2(\mathcal{U},\mathcal{S}^1) & \to & \\
\uparrow & & \uparrow & & \\
C^1(\mathcal{U},\mathcal{S}^o) & \to & C^1(\mathcal{U},\mathcal{S}^1) & \to & \\
\uparrow & & \uparrow & & \\
C^o(\mathcal{U},\mathcal{S}^o) & \to & C^o(\mathcal{U},\mathcal{S}^1) & \to & C^o(\mathcal{U},\mathcal{S}^2) \to \ldots
\end{array}$$

In this commutative diagram we see that the rows are exact except for the terms which appear when we compute the kernels of the maps from $C^p(\mathcal{U}, \mathcal{S}^o) \to C^p(\mathcal{U}, \mathcal{S}^1)$ and these kernels are the groups $C^p(\mathcal{U}, \mathcal{S})$.

Finally, if we restrict the attention to a single column we are computing the δ-cohomology of the complex

$0 \to C^o(\mathcal{U}, \mathcal{S}^q) \to C^1(\mathcal{U}, \mathcal{S}^q)$ ... whose cohomology groups are $\{H^p(\mathcal{U}, \mathcal{S}^q)\}$ and since $\{\mathcal{S}^q\}$ are fine sheaves only the group $H^o(\mathcal{U}, \mathcal{S}^q) = \Gamma(X, \mathcal{S}^q)$ appears.

Summing up, computing the cohomology in the rows gives the string $C^o(\mathcal{U}, \mathcal{S})$, $C^1(\mathcal{U}, \mathcal{S})$, ... along the first column. Computing the cohomology in the columns gives the string

$\Gamma(X, \mathcal{S}^o)$, $\Gamma(X, \mathcal{S}^1)$, ... along the lowest row.

At this stage an easy diagram chasing proves that the cohomology of the two complexes

$0 \to C^o(\mathcal{U}, \mathcal{S}) \to C^1(\mathcal{U}, \mathcal{S}) \to \dots$ and $0 \to \Gamma(X, \mathcal{S}^o) \to \Gamma(X, \mathcal{S}^1) \to \dots$

are equal. By definition this means that

$H^p(\mathcal{U}, \mathcal{S})$ are the cohomology groups which appear in the complex $0 \to \Gamma(X, \mathcal{S}^o) \to \Gamma(X, \mathcal{S}^1) \to \dots$ and finally Lemma 4.6. shows that these cohomology groups are $H^p(X, \mathcal{S})$.

5. Some Final Remarks

During the last part of the proof of Leray's Theorem we encounter a special case which deals with the hypercohomology of a double complex and using the machinery based upon spectral sequences we could have given more sophisticated proofs of the results in Section 4. We refer to other notes for results about spectral sequences. Let us only remark that this more powerful approach is necessary when we begin to consider complexes

<u>of sheaves</u> $0 \to \mathcal{S}^0 \to \mathcal{S}^1 \to \mathcal{S}^2 \to \ \ldots$ and try to understand how their cohomology groups are related to each other. It turns out that the natural device in such a situation is the following: If $\mathfrak{U}$ is an open covering of X we get the double complex

$$\begin{tikzcd}
C^2(\mathfrak{U},\mathcal{S}^0) \arrow[r] & {} & & \\
C^1(\mathfrak{U},\mathcal{S}^0) \arrow[r] \arrow[u] & {} & & \\
C^0(\mathfrak{U},\mathcal{S}^0) \arrow[r] \arrow[u] & C^0(\mathfrak{U},\mathcal{S}^1) \arrow[r] \arrow[u] & C^0(\mathfrak{U},\mathcal{S}^2) \arrow[r] & \cdots
\end{tikzcd}$$

and in this double complex we can compute the so called <u>hypercohomology</u> along the diangonals and these groups are called the hypercohomology groups of the complex of sheaves. Then we can recaputure the hypercohomology by various spectral sequences and in favourable cases this gives a good information about the complex

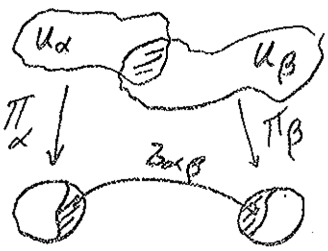
$$0 \to \Gamma(X,\mathcal{S}^0) \to \Gamma(X,\mathcal{S}^1) \to \ \ldots .$$

Chapter 2

Let us first define compact Riemann surfaces. A compact Riemann surface is a compact and connected topological space X which is equipped with an open covering $\{ U_\alpha \}$ satisfying the following :

To each α there exists a homeomorphism $\pi_\alpha : U_\alpha \to \Delta_\alpha$ where Δ_α is some open disc in the complex plane and $\{ U_\alpha, \pi_\alpha, {}_\alpha \}$ satisfy the following condition when two sets U_α and U_β have a non-empty intersection:

Looking at the picture below



we get the map $\sigma_{\alpha\beta}$ from $\pi_\alpha(U_\alpha \cap U_\beta)$ into $\pi_\beta(U_\alpha \cap U_\beta)$ and we require that $\sigma_{\alpha\beta}$ is a biholomorphic mapping between these two open subsets of C^1.

We say that $\{ U_\alpha, \pi_\alpha \}$ is an atlas which defines the complex analytic strucutre on X and the open sets $\{ U_\alpha \}$ are called charts.

Of course, since X is compact we can assume that $\{ U_\alpha \}$ is a finite family of open sets.

From now on this atlas is fixed and we put $\mathfrak{U} = \{U_\alpha\}$

1.1. The sheaf $\mathcal{O}$. Let U be an open set in X. A function f in U is holomorphic if the functions $f_\alpha(z) = f(\pi_\alpha^{-1}(z))$ are holomorphic on $\pi_\alpha(U \cap U_\alpha)$ for all α. In particular we observe that if $g_\alpha \in \mathcal{O}(\Delta_\alpha)$ is given for some α and if we define $f(x) = g_\alpha(\pi_\alpha(x))$ for all $x \in U_\alpha$, then f is holomorphic in the open set U_α. So in each chart U_α there exist holomorphic functions.

Let $\Gamma(U, \mathcal{O})$ denote the set of holomorphic functions in an open subset

U of X. Hence $\Gamma(U_\alpha, \mathcal{O}) \cong \mathcal{O}(\Delta_\alpha)$ for every chart.

We get the sheaf $\mathcal{O}$ of holomorphic functions on X where $\Gamma(U, \mathcal{O})$ are the sections over open sets U.

In the same way we get the sheaf $\mathcal{M}$ of meromorphic functions on X. A section $f \in \Gamma(U, \mathcal{M})$ is a function on the open set U such that the functions $f_\alpha(z) = f(\pi_\alpha^{-1}(z))$ are meromorphic on $\pi_\alpha(U \cap U_\alpha)$ for all α.

Let us observe the following

Lemma 1.2. $\Gamma(X, \mathcal{O}) = C$

Proof Let f be a globally defined holomorphic function on X. Since X is compact and f is a continuous function, it follows that the absolute value $|f|$ attains a maximum at some point $x \in X$. Here $x \in U_\alpha$ for some chart and the maximum principle for analytic functions in the open disc Δ_α shows that the function $f_\alpha(z) = f(\pi_\alpha^{-1}(z))$ is constant in Δ_α, which means that f is constant on the whole chart U_α. Since X is connected we can pass to other charts and conclude that f is a constant on X.

The question arises if there exist non-constant globally defined meromorphic functions on X. This is not at all obvious and we give the proof in Section 3 below.

1.3. X as a differentiable manifold

The complex z-plane can be considered as the real 2-dimensional (x,y)-space, where $z = x + iy$. Since holomorphic functions are differentiable the atlas $\{U_\alpha, \pi_\alpha\}$ defines a differentiable structure on X. Hence we can speak about (infinitely) differentiable functions on X, and of differential forms of order 1 and 2, where X is a compact and 2-dimensional differentiable manifold.

In particular we can introduce the following objects.

1.3.1. $\mathcal{E}$ = The sheaf of complex-valued C^∞-functions on X

1.3.2. $\mathcal{E}^1$ = The sheaf of differential 1-forms on X with coefficients in $\mathcal{E}$.

Remark Consider a chart U_α where we define $z_\alpha = \pi_\alpha^{-1}(z)$ so that z_α is a holomorphic function in U_α and gives the complex coordinates in U_α. We can write $z_\alpha = x_\alpha + iy_\alpha$ where x_α and y_α are real-valued and then the differentials dx_α and dy_α is a basis for the 1-forms on U_α. This means that if $\varphi \in \Gamma(U_\alpha, \mathcal{E}^1)$ then $\varphi = f_\alpha dx_\alpha + g_\alpha dy_\alpha$ where f_α and g_α belong to $C^\infty(U_\alpha)$.

1.3.3. The sheaves $\mathcal{E}^{0,1}$ and $\mathcal{E}^{1,0}$.

The complex analytic structure on X enable us to decompose 1-forms using the differentials $dz_\alpha = dx_\alpha + idy_\alpha$ and $d\bar{z}_\alpha = dx_\alpha - idy_\alpha$ in the various charts U_α. This gives the sheaf $\mathcal{E}^{1,0}$ where a section $\varphi \in \Gamma(U, \mathcal{E}^{1,0})$ is a differential 1-form such that in each intersection $U \cap U_\alpha$ we get $\varphi = f_\alpha dz_\alpha = f_\alpha(dx_\alpha + idy_\alpha)$ for some $f_\alpha \in C^\infty(U \cap U_\alpha)$.

Similarly $\mathcal{E}^{0,1}$ is the sheaf whose sections in the charts U_α are expressed by $g_\alpha d\bar{z}_\alpha$

In this way $\mathcal{E}^1 = \mathcal{E}^{1,0} \oplus \mathcal{E}^{0,1}$ and the usual exterior differential $d : \mathcal{E} \to \mathcal{E}^1$ splits into $\partial : \mathcal{E} \to \mathcal{E}^{1,0}$ and $\bar{\partial} : \mathcal{E} \to \mathcal{E}^{0,1}$

More precisely, in a chart U_α we get $df = \partial f/\partial x_\alpha \, dx_\alpha + \partial f/\partial y_\alpha \, dy_\alpha$ and we can write $df = \partial f + \bar{\partial} f$, where

$$\partial f = \partial f/\partial z_\alpha \, dz_\alpha \quad \text{and} \quad \bar{\partial} f = \partial f/\partial \bar{z}_\alpha \, d\bar{z}_\alpha$$

The condition that f is holomorphic is that $\bar{\partial} f = 0$, i.e. this is expressed by the Cauchy-Riemann equations.

1.3.4. The sheaf $\mathcal{O}^{1,0}$.

$\mathcal{E}^{1,0}$ contains a subsheaf $\mathcal{O}^{1,0}$ whose sections in a chart U_α have the form $h_\alpha\, dz_\alpha$ where $h_\alpha \in \Gamma(U_\alpha, \mathcal{O})$. $\mathcal{O}^{1,0}$ is the sheaf of holomorphic 1-forms. The global sections $\Gamma(X, \mathcal{O}^{1,0})$ is the set of abelian differentials and they play an important role later on.

1.4. The Dolbeault Isomorphism

Let us first recall that C^∞ functions on manifolds admit partitions of the unity. It follows easily that $\mathcal{E}, \mathcal{E}^1, \mathcal{E}^{1,0}, \mathcal{E}^{0,1}$ and $\mathcal{E}^2$ are fine sheaves. In particular their cohomology groups vanish when $p > 0$.

Let us also recall that the $\bar\partial$-equation can be solved in open discs in the complex plane. This means that if $p(z) \in C^\infty(\Delta)$ then there exists some $g \in C^\infty(\Delta)$ such that $\partial g/\partial \bar z = p$ which means that $\bar\partial g = p\, d\bar z$.

We leave out the easy and wellknown proof.

Passing to the Riemann surface X, the similar conclusion holds in the charts of X and hence we get the following exact sequence of sheaves:

$$0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\bar\partial} \mathcal{E}^{0,1} \to 0$$

If we consider the global sections we get the exact sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{E}) \xrightarrow{\bar\partial} \Gamma(X, \mathcal{E}^{0,1}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{E}) \to \dots$$

and since $H^1(X, \mathcal{E}) = 0$ we conclude that

$H^1(X, \mathcal{O}) = \Gamma(X, \mathcal{E}^{0,1}) / \bar\partial(\Gamma(X, \mathcal{E}))$. This is the Dolbeult isomorphism.

1.5. Holomorphic Line bundles

Let $\mathcal{O}^*$ be the sheaf of holomorphic functions on X which are nowhere-vanishing. This means that a section $f \in \Gamma(U, \mathcal{O}^*)$ is a holomorphic function in U which never is zero.

The group structure in the sheaf $\mathcal{O}^*$ arises when we take the products fg of two sections.

We get a sheaf homomorphism from $\mathcal{O}$ into $\mathcal{O}^*$ using the exponential map which to every holomorphic function g gives $e^{2\pi i g}$ in $\mathcal{O}^*$.

Here $e^g = 1$ in $\mathcal{O}^*$ if and only if g is locally constant with integer-values.

So if $\underline{Z}$ is the constant sheaf whose stalks are the groups of integers, then we get the exact sequence of sheaves :

$$0 \to \underline{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$$

Before we continue we observe

Lemma 1.5.1. Let $\mathcal{U} = \{ U_\alpha \}$ be the covering of X of charts $\{ U_\alpha \}$. Then $\mathcal{U}$ is a Leray covering with respect to $\mathcal{O}$ and with respect to $\mathcal{O}^*$.

Proof Each non-empty intersection $U_{\alpha_0 \ldots \alpha_p}$ is biholomorphic with an open and simply connected subset of the complex plane. By Riemann's Mapping Theorem these sets are biholomorphic with an open disc Δ. So it is sufficient to prove that $H^p(\Delta, \mathcal{O}) = H^p(\Delta, \mathcal{O}^*) = 0$ for all $p > 0$.

To prove this we first observe that the Dolbeault isomorphism and the existence of solutions to the $\bar\partial$-equation in a disc implies that

$H^1(\mathcal{O}, \Delta) = 0$. Similarly, we have the whole exact sequence

$0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\bar\partial} \mathcal{E}^{1,0} \to 0$ where $\mathcal{E}$ and $\mathcal{E}^{1,0}$ are fine sheaves and it follows that the cohomology groups $\{ H^p(X, \mathcal{O}) \}$ are the cohomology groups in the complex $0 \to \Gamma(X, \mathcal{E}) \to \Gamma(X, \mathcal{E}^{0,1}) \to 0$. This implies that $H^p(X, \mathcal{O}) = 0$ for all $p \geq 2$ and of course we also get $H^p(\Delta, \mathcal{O}) = 0$ for all $p \geqslant 2$.

Next, since a disc Δ is contractible as a topological space, it follows that $H^p(\Delta, \underline{Z}) = 0$ for all $p > 0$.

The exact sequence $0 \to \underline{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$ gives the long exact sequence $0 \to \Gamma(\Delta, \underline{Z}) \to \Gamma(\Delta, \mathcal{O}) \to \Gamma(\Delta, \mathcal{O}^*) \to H^1(\Delta, \underline{Z}) \to H^1(\Delta, \mathcal{O}) \to$
$\to H^1(\Delta, \mathcal{O}^*) \to \ldots$

and we conclude that $H^p(\Delta, \mathcal{O}^*) = 0$ for all $p > 0$.

When Δ is replaced by X we get the exact sequence

$$0 \to \Gamma(X, \underline{Z}) \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{O}^*) \to H^1(X, \underline{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to$$
$$\to H^2(X, \underline{Z}) \to H^2(X, \mathcal{O}) \quad \ldots.$$

We have already proved that $H^2(X, \mathcal{O}) = 0$ and that $\Gamma(X, \mathcal{O}) = C$ and similarly $\Gamma(X, \mathcal{O}^*) = C^*$ and of course $\Gamma(X, \underline{Z}) = \underline{Z}$ and the sequence $0 \to Z \to C \xrightarrow{\exp} C^* \to 0$ is exact. We conclude

Proposition 1.5.2. There exists an exact sequence

$$0 \to H^1(X, \underline{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \underline{Z}) \to 0$$

We shall return to this exact sequence in section 3.

1.5.3. The sheaves $\mathcal{O}(\xi)$

Consider an element $\xi \in H^1(X, \ ^*)$. Since $\mathcal{U}$ is a Leray covering with respect to $\mathcal{O}^*$ we can represent ξ by a cocycle in $Z^1(\mathcal{U}, \mathcal{O}^*)$. This means that there exists a family $\{ \xi_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*) \}$ satisfying

$\xi_{\alpha\sigma} = \xi_{\alpha\beta}\, \xi_{\beta\sigma}$ in each intersection $U_\alpha \cap U_\beta \cap U_\sigma$.

The cocycle $\{ \xi_{\alpha\beta} \}$ is unique module an element in $B^1(\mathcal{U}, \mathcal{O}^*)$. This means that if $\{ \xi'_{\alpha\beta} \}$ is another cocycle representing the cohomology class ξ, then there exist functions $\{ g_\alpha \in \Gamma(U_\alpha, \mathcal{O}^*) \}$ such that

$\xi'_{\alpha\beta} = (g_\alpha/g_\beta)\xi_{\alpha\beta}$ for all α and β.

When $\{\xi_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ is given we define a sheaf $\mathcal{O}(\xi)$ as follows:

If U is an open subset of X then a section in $\Gamma(U, \mathcal{O}(\xi))$ consists of a family $\{ f_\alpha \in \Gamma(U \cap U_\alpha, \mathcal{O})\}$ satisfying : $f_\beta = \xi_{\beta\alpha} f_\alpha$ in each non-empty intersection $U \cap U_\alpha \cap U_\beta$.

It is easily seen that the sheaf $\mathcal{O}(\xi)$ is unique, up to an isomorphism. In fact, if $\{\xi'_{\alpha\beta}\}$ is another coclycle representing ξ and if $\mathcal{O}(\xi')$ is the sheaf defined by this cocycle then we see that if

$$f' = \{ f'_\alpha \} \in \Gamma(U, \mathcal{O}(\xi')) \quad \text{then } f = \{ f'_\alpha / g_\alpha \} \in \Gamma(U, \mathcal{O}(\xi))$$

where $\xi'_{\alpha\beta} = (g_\alpha/g_\beta)\xi_{\alpha\beta}$ and this is a sheaf isomorphism.

Summing up, if $\xi \in H^1(X, \mathcal{O}^*)$ then we get a sheaf $\mathcal{O}(\xi)$. Of course, the sheaf $\mathcal{O}(\xi)$ is locally isomorphic to $\mathcal{O}$. In fact, we see that $\mathcal{O}(\xi) \simeq \mathcal{O}$ over each chart U_α and we can say that $\mathcal{O}(\xi)$ is a locally free sheaf of $\mathcal{O}$-modules of rank one.

1.5.4. <u>The canonical line bundle</u> $\mathcal{K}$.

The sheaf $\mathcal{O}^{1,0}$ from Section 1.3.4. is of the form $\mathcal{O}(\mathcal{K})$ for some $\mathcal{K} \in H^1(X, \mathcal{O}^*)$. Let us explain this. In each chart U_α we get a complex coordinate function dz_α. If $U_\alpha \cap U_\beta$ is non-empty then dz_α or dz_β can be used as a basis for holomorphic differential forms in U_α U_β. In particular we get $dz_\beta = \eta_{\alpha\beta}\, dz_\alpha$ where $\eta_{\alpha\beta} \in \Gamma(U_\alpha \cap U_\beta, \mathcal{O}^*)$ and obviously $\{\eta_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and hence this cocycle represents a cohomology class $\mathcal{K}$.

By definition a section $f \in \Gamma(U, \mathcal{O}(\mathcal{K}))$ consists of a collection $\{ f_\alpha \in \Gamma(U \cap U_\alpha, \mathcal{O})$ satisfying $f_\alpha = \eta_{\alpha\beta} f_\beta$ in $U_\alpha \cap U_\beta \cap U \}$

These equations give

$f_\beta dz_\beta = f_\beta \eta_{\alpha\beta} dz_\alpha = f_\alpha dz_\alpha$ and hence there exists a holomorphic 1-form ω on U whose restriction to $U \cap U_\alpha$ is $f_\alpha dz_\alpha$ for all α.

In this way $\Gamma(U, \mathcal{O}^{1,0}) \cong \Gamma(U, \mathcal{O}(\mathcal{K}))$.

$\mathcal{O}(\mathcal{K}) = \mathcal{O}^{1,0}$ is called the canonical line bundle on X.

2. The Duality Theorem

Let $\{ \xi_{\alpha\beta} \} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ represent a sheaf $\mathcal{O}(\xi)$. We can also define the sheaf $\mathcal{E}(\xi)$ whose sections over an open set U consist of families $\{ f_\alpha \in C^\infty(U \cap U_\alpha) : f_\alpha = \xi_{\alpha\beta} f_\beta \text{ in } U \cap U_\alpha \cap U_\beta \}$

Then $\mathcal{O}(\xi)$ is a subsheaf of $\mathcal{E}(\xi)$. Let us also observe that if $f = \{ f_\alpha \in C^\infty(U \cap U_\alpha) \} \in \Gamma(U, \mathcal{E}(\xi))$, then we can consider the (o,1)-forms $\{ \bar\partial f_\alpha \}$ and since $\xi_{\alpha\beta}$ are holomorphic we see that

$$\bar\partial f_\alpha = \xi_{\alpha\beta} \bar\partial f_\beta \quad \text{for all } \alpha \text{ and } \beta.$$

Hence we can say that $\bar\partial$ defines a sheaf homomorphism from $\mathcal{E}(\xi)$ into $\mathcal{E}^{0,1}(\xi)$ where $\mathcal{E}^{0,1}(\xi)$ has sections given by a family $\{ \omega_\alpha \in \Gamma(U \cap U_\alpha, \mathcal{E}^{0,1}) : \omega_\alpha = \xi_{\alpha\beta} \omega_\beta \text{ in } U \cap U_\alpha \cap U_\beta \}$

Both $\mathcal{E}(\xi)$ and $\mathcal{E}^{0,1}(\xi)$ are locally free sheaves of $\mathcal{E}$-modules and in particular they are fine sheaves. We get (exactly as in 1.4.) the exact sequence of sheaves $0 \to \mathcal{O}(\xi) \to \mathcal{E}(\xi) \to \mathcal{E}^{0,1}(\xi) \to 0$ and the following isomorphism :

$$H^1(X, \mathcal{O}(\xi)) \cong \Gamma(X, \mathcal{E}^{0,1}(\xi)) / \bar\partial(\Gamma(X, \mathcal{E}(\xi)))$$

At this stage we observe that $\Gamma(X, \mathcal{E}^{0,1}(\xi))$ has a natural structure as a topological space. Indeed, this is a linear space and using the usual topology on C^∞-functions, $\Gamma(X, \mathcal{E}^{0,1}(\xi))$ is a Frechet space ~~and the resulting quotient space $H^1(X, \mathcal{O}(\xi))$ also exists.~~

It is not obvious that $\bar{\partial}(\Gamma(X,\mathcal{E}(\xi)))$ is a closed subspace. But it follows if we can prove

<u>Theorem</u> 2.1. Let $\xi \in H^1(X,\mathcal{O}^*)$. Then $H^0(X, \mathcal{O}(\xi))$ and $H^1(X, \mathcal{O}(\xi))$ are finite dimensional complex vector spaces.

<u>Proof</u> The idea is to use some wellknown facts about holomorphic functions. Recall that if $\{ f_v \}$ is a bounded sequence of holomorphic functions defined in some open subset of C^1, then $\{ f_v \}$ contains a subsequence which is uniformly convergent over relatively compact subsets.

In particular we get the following

<u>Sublemma</u> 1. Let $V \ll V'$ be two open subsets of C^1, i.e. V is a relatively compact subset of V'. Let $\Gamma_0(V,\mathcal{O})$ denote the set of holomorphic functions f(z) in V which are square integrable, i.e. $\int_V |f(z)|^2 \, dxdy < \infty$. $\Gamma_0(V,\mathcal{O})$ and $\Gamma_0(V',\mathcal{O})$ are both Hilbert spaces and the restriction mapping from $\Gamma_0(V,\mathcal{O})$ into $\Gamma_0(V',\mathcal{O})$ is a bounded linear operator which maps bounded subsets of $\Gamma_0(V,\mathcal{O})$ into relatively compact subsets of $\Gamma_0(V',\mathcal{O})$. In particular, if L is some closed linear subspace of $\Gamma_0(V',\mathcal{O})$ which is contained in the image of $\Gamma_0(V,\mathcal{O})$, then basic functional analysis proves that L is a finite dimensional subspace of $\Gamma_0(V',\mathcal{O})$

It remains only to see how this Sublemma can be applied to the present situation.

The cohomology class ξ can be represented by a cocycle $\{ \xi_{\alpha\beta} \}$ in $Z^1(\mathcal{U},\mathcal{O}^*)$ where $\mathcal{U}$ is some covering of X with charts.

Now we can consider $C_0(\mathcal{U},\mathcal{O}(\xi))$ = the 0-cochains represented by families $\{ g_\alpha \in \Gamma_o(V_\alpha, \mathcal{O}) : g_\alpha = \xi_{\alpha\beta}\, g_\beta \text{ in } V_\alpha \cap V_\beta \}$

Hence $C_o(\mathcal{U}, \mathcal{O}(\xi))$ is the set of square integrable cochains of with coefficients in $\mathcal{O}(\xi)$.

Similarly we consider the set $C_o^{\,1}(\mathcal{U}, \mathcal{O}(\xi))$ of square integrable 1-cochains of $\mathcal{U}$ with coefficients in $\mathcal{O}(\xi)$.

Since the coboundary operator involves restrictions and finite summations (Of course $\mathcal{U} = \{ V_\alpha \}$ is taken as a finite covering of X) it is clear that δ maps $C_o^{\,p}(\mathcal{U}, \mathcal{O}(\xi))$ into $C_o^{p+1}(\mathcal{U}, \mathcal{O}(\xi))$ and we get the corresponding " suqare-integrable cohomology groups" defined by

$$H_o^p(\mathcal{U}, \mathcal{O}(\xi)) = Z_o^{\,p}(\mathcal{U}, \mathcal{O}(\xi))/ \delta(C_o^{\,p-1}(\mathcal{U}, \mathcal{O}(\xi)))$$

The inclusions $C_o^{\,p} (\mathcal{U}, \mathcal{O}(\xi)) \subset C^p(\mathcal{U}, \mathcal{O}(\xi))$ induces a homomorphism from $H_o^{\,p}(\mathcal{U}, \mathcal{O}(\xi))$ into $H^p(\mathcal{U}, \mathcal{O}(\xi))$ for each p .

With these notations we can prove

<u>Sublemma</u> 2. Let $\mathcal{V} = \{ V_\alpha \}$ and $\Omega = \{ \Omega_\alpha \}$ be two open coverings of X by charts, where $\mathcal{V}$ is a refinement of Ω such that $\bar{V}_\alpha \subset \Omega_\alpha$ for all α . Then $H_o^{\,p}(\mathcal{V}, \mathcal{O}(\xi)) = H^p(\mathcal{V}, \mathcal{O}(\xi))$ for p = 0 and p = 1

<u>Proof</u> Let us first observe that the map i^* : $H_o^{\,p}(\mathcal{V}, \mathcal{O}(\xi))$ into $H^p(\mathcal{V}, \mathcal{O}(\xi))$ is an injection for p = 0 and p = 1. When p = 0 this is entirely obvious because $H_o^0(\mathcal{V}, \mathcal{O}(\xi)) = \Gamma_o(X, \mathcal{O}(\xi))$ which appears as a subspace of $\Gamma(X, \mathcal{O}(\xi))$. If p = 1 we can argue as follows:

Select a cocycle $\{ f_{\alpha\beta} \} \in Z_o^{\,1}(\mathcal{V}, \mathcal{O}(\xi))$ and suppose that $\{ f_{\alpha\beta} \}$ is cohomologous to zero in $H^1(\mathcal{V}, \mathcal{O}(\xi))$. This means that

$f_{\alpha\beta} = f_\alpha - f_\beta$, where $(f_\alpha) \in C^1(\mathcal{V}, \mathcal{O}(\xi))$ and it suffices to prove that each f_α is square integrable over V_α, for then $\{ f_\alpha \}$ belongs to $C_o^{\,o}(\mathcal{V}, \mathcal{O}(\xi))$ and $\{ f_{\alpha\beta} \} = \delta(\{ f_\alpha \})$ has the image zero in $H_o^{\,1}(\mathcal{V}, \mathcal{O}(\xi))$.

To prove this we select an open set V_α and consider some boundary point $p \in \partial V_\alpha$. Now $p \in V_\beta$ for some β and a neighborhood (U) of p stays in U_β and we have $f_{\alpha\beta} = f_\alpha - f_\beta$ in $\mathcal{U} \cap V_\alpha$.

Here both $f_{\alpha\beta}$ and f_β are square integrable in $V_\alpha \cap U$ (if U is chosen so that $U \subset\subset V_\beta$) and hence f_α is locally square integrable in a neighborhood of each boundary point of V_α. Since ∂V_α is compact, Heine-Borels Lemma implies that $f_\alpha \in \Gamma_o(V_\alpha, \mathcal{O})$, as required.

<u>Proof continued</u> Since $\mathcal{V}$ and Ω are Leray coverings with respect to the sheaf $\mathcal{O}(\xi)$ we can apply Leray's Theorem and conclude that

$$H^p(\mathcal{V}, \mathcal{O}(\xi)) \cong H^p(\Omega, \mathcal{O}(\xi)) = H^p(X, \mathcal{O}(\xi))$$

The restriction maps $C^j(\Omega, \mathcal{O}(\xi))$ into $C_o^{\,j}(\mathcal{V}, \mathcal{O}(\xi))$ so this isomorphism implies that a cohomology class in $H^p(\mathcal{V}, \mathcal{O}(\xi))$ can be represented by the restriction of some cocycle in $Z^p(\Omega, \mathcal{O}(\xi))$ to $\mathcal{V}$, and this restriction belongs to $Z_o(\mathcal{V}, \mathcal{O}(\xi))$ and we conclude that the map from $H_o^{\,p}(\mathcal{V}, \mathcal{O}(\xi))$ into $H^p(X, \mathcal{O}(\xi))$ is surjective. This proves Sublemma 2.

<u>Final part of the proof</u>

We can choose 3 Leray coverings $\mathcal{V} << \mathcal{V}' << \mathcal{V}''$ where the two pairs $(\mathcal{V}, \mathcal{V}')$ and $(\mathcal{V}', \mathcal{V}'')$ satisfy the conditions in Sublemma 2.

Hence Sublemma 2 gives $H^p(X, \mathcal{O}(\xi)) = H_o^{\,p}(\mathcal{V}, \mathcal{O}(\xi)) = H_o^{\,p}(\mathcal{V}', \mathcal{O}(\xi))$ for each p = 0 and 1. Finally, the restriction map from $Z_o^{\,p}(\mathcal{V}' \mathcal{O}(\xi))$ into $Z_o^{\,p}(\mathcal{V}, \mathcal{O}(\xi))$ sends bounded sets into relatively compact sets and at this stage we leave the final details which uses functional analysis and proves that $H^p(X, \mathcal{O}(\xi))$ are finite dimensional.

2.2. <u>The Duality Theorem</u>. Let $\xi \in H^1(X, \mathcal{O}^*)$. We get the sheaf $\mathcal{O}(\xi)$ and we have proved that $H^p(X, \mathcal{O}(\xi))$ are finite dimensional vector spaces. We can study their dual spaces and the result we prove is this.

<u>Theorem</u> 2.2. $(H^1(X, \mathcal{O}(\xi))^* \cong H^o(X, \mathcal{O}(\xi^{-1} \mathcal{K}))$ where we recall that $\mathcal{K}$ is the canonical line bundle.

<u>Proof</u> Since $H^1(X, \mathcal{O}(\xi)) = \Gamma(X, \mathcal{E}^{o,1}(\xi)) / \bar{\partial}(\Gamma(X, \mathcal{E}(\xi)))$ we begin to study continuous linear forms on the Frechet space $\Gamma(X, \mathcal{E}^{o,1}(\xi))$ which vanish on the subspace $\bar{\partial}(\Gamma(X, \mathcal{E}(\xi)))$.

Let L be such a linear form. We begin to consider the restriction of L to certain subspaces of $\Gamma(X, \mathcal{E}^{o,1}(\xi))$ arising as follows:

In general, ξ is represented by the cocycle $\{ \xi_{\alpha\beta} \} \in Z^1(\mathcal{U}, \mathcal{O}^*)$ and if U_α is one of these charts and if $f \in C_o^\infty(U_\alpha)$ then f gives a globally defined section $\alpha(F)$ in $\Gamma(X, \mathcal{E}^{o,1}(\xi))$ as follows:

First, since f has a compact support in U_α, the (o,1)-form $fd\bar{z}_\alpha$ in U_α extends to a globally defined (o,1)-form on X when we define its value to be zero outside U_α.

Now we can put $w_\alpha = fd\bar{z}_\alpha$ and for the remaining indices β we can restrict the 1-form $fd\bar{z}_\alpha$ to the open set U_β and since $\mathrm{supp}(f) \subset U_\alpha$ we see that $\mathrm{supp}(f\, d\bar{z}_\alpha) \subset U_\alpha \cap U_\beta$ inside U_β which means that we can multiply by the holomorphic function $\xi_{\beta\alpha}$ and get the 1-form

$$w_\beta = \xi_{\beta\alpha}\, fd\bar{z}_\alpha = \xi_{\beta\alpha}\, w_\alpha \quad \text{for all } \beta \neq \alpha.$$

The definition of the sheaf $\mathcal{E}^{o,1}(\xi)$ shows that the family of all these (o,1)-forms gives a global section $\alpha(F) \in \Gamma(X, \mathcal{E}^{o,1}(\xi))$

Hence $f \to \alpha(F)$ is a linear map and this is a continuous map form the Frechet space $C_o^{\infty}(U_\alpha)$ into $\Gamma(X, \mathcal{E}^{o,1}(\xi))$. In fact, the Frechet space topology on the global sections of the fine sheaf $\mathcal{E}^{o,1}(\xi)$ is defined by these conditions for all α.

The linear form L therefore gives a linear form L_α on $C_o^{\infty}(U_\alpha)$ defined by $L(\alpha(F)) = L_\alpha(f)$ for all $f \in C_o^{\infty}(U_\alpha)$. This means that there exists a distribution μ_α on U_α such that $L_\alpha(f) = \langle f, \mu_\alpha \rangle$

Now we observe that if $g \in C_o^{\infty}(U_\alpha)$ and if $f = \partial g/\partial \bar{z}_\alpha$ so that $f d\bar{z}_\alpha = \bar{\partial} g$, then $\alpha(F) = \bar{\partial}\alpha(G)$ where $G = \{ g_\beta = \xi_{\beta\alpha} g \} \in \Gamma(X, \mathcal{E}(\xi))$ and since $L = 0$ on $\bar{\partial}(\Gamma(X, \mathcal{E}(\xi)))$, we conclude that

$\langle \partial g/\partial\bar{z}_\alpha, \mu_\alpha \rangle = 0$ for all $g \in C_o^{\infty}(U_\alpha)$. This means that the distribution μ_α satisfies the $\bar{\partial}$-equation in the chart U_α. Recall that the $\bar{\partial}$-equation is elliptic. Hence Weyl's Lemma implies that μ_α is smooth and the Cauchy-Riemann equation gives a holomorphic function g_α on U_α such that $L_\alpha(f) = \int f\, g_\alpha\, dz_\alpha \wedge d\bar{z}_\alpha$ for all $f \in C_o^{\infty}(U_\alpha)$

Summing up, the linear form L gives a collection of holomorphic functions $\{ g_\alpha \in \Gamma(U_\alpha, \mathcal{O}) \}$ and it remains to see how these functions are related to each other in non-empty intersections $U_\alpha \cap U_\beta$.

To see this we begin with some $f \in C_o^{\infty}(U_\alpha \cap U_\beta)$. We get the element $\alpha(F)$ in $\Gamma(X, \mathcal{E}^{o,1}(\xi))$ and now we try to find some $f' \in C_o^{\infty}(U_\alpha \cap U_\beta)$ satisfying : $\beta(F') = \alpha(F)$

Looking at the construction of $\alpha(F)$, and $\beta(F')$, where $\beta(F')$ arises when f' is considered as an element in $C_0^{\infty}(U_\beta)$, we see that

$\alpha(F) = \beta(F')$ holds if $f d\bar{z}_\alpha = \xi_{\alpha\beta} f' d\bar{z}_\beta$ holds in $U_\alpha \cap U_\beta$

Recall that $dz_\beta = \eta_{\alpha\beta} dz_\alpha$ which gives $d\bar{z}_\beta = \overline{\eta_{\alpha\beta}}\, d\bar{z}_\alpha$ and hence we put $f = \xi_{\alpha\beta}\, \overline{\eta_{\alpha\beta}}\, f'$

The equality $\alpha(F) = \beta(F')$ gives

$L_\alpha(f) = L_\beta(f')$ which means that the two double integrals

$$\int f\, g_\alpha\, dz_\alpha \wedge d\bar{z}_\alpha = \int f'\, g_\beta\, dz_\beta \wedge d\bar{z}_\beta$$

Here $dz_\beta \wedge d\bar{z}_\beta = |\eta_{\alpha\beta}|^2\, dz_\alpha \wedge d\bar{z}_\alpha$ and since $f' = (\xi_{\alpha\beta})^{-1} (\overline{\eta_{\alpha\beta}})^{-1} f$

we conclude that $f\, g_\alpha\, dz_\alpha \wedge d\bar{z}_\alpha = f\, g_\beta\, (\xi_{\alpha\beta})^{-1} \eta_{\alpha\beta}\, dz_\alpha \wedge d\bar{z}_\alpha$

Since this equality holds for all $f \in C_0^{\infty}(U_\alpha \cap U_\beta)$ we see that

$$g_\alpha = (\xi_{\alpha\beta})^{-1} \eta_{\alpha\beta}\, g_\beta \quad \text{for all } \alpha \text{ and } \beta.$$

But this means that $\{ g_\alpha \}$ defines a section in $\Gamma(X, \mathcal{O}(\xi^{-1} \mathcal{K})) = H^0(X, \mathcal{O}(\xi^{-1}\mathcal{K}))$. Conversely, the arguments above show that such global sections enable us to construct linear forms on $\Gamma(X, \mathcal{E}^{0,1}(\xi))$ and they vanish on the subspace $\bar{\partial}(\Gamma(X, \mathcal{E}(\xi)))$. This completes the proof.

2.3. Divisors and their Line Bundles

A divisor on X is a finite collection $\{ p_i, v_i \}$ where $\{ p_1 \dots p_s \}$ is a finite set of points on X and $v_1 \dots v_s$ are integers. We write

$$\delta = \Sigma \; v_i \zeta_{p_i}$$

Hence the divisors is a free abelian group where the points on X give a free basis. We can add divisors and so on.

The degree of a divisor $\Sigma = \Sigma \; v_i \zeta_{p_i}$ is defined as the integer $\Sigma \; v_i$ and this integer is denoted by $|\delta|$.

2.3.1. The divisor of a meromorphic function

If f is a meromorphic function on X, the zeros and the poles of f is a discrete subset. Since X is compact there are only finitely many zeros and poles and we get the divisor $\delta = \mathrm{div}(f) = \Sigma \; v_i \zeta_{p_i}$ where $v_i > 0$ when f has a zero of order v_i at the point p_i, while $v_i < 0$ if f has a pole of order v_i at the point p_i.

Divisors arising from globally defined meromorphic functions on X are called principal divisors. The principal divisors form a subgroup because $\mathrm{div}(fg) = \mathrm{div}(f) + \mathrm{div}(g)$ when f and $g \in \Gamma(X, \mathcal{M}^*)$, where $\mathcal{M}^*$ is the sheaf of meromorphic functions on X which are not identically zero.

2.3.2. The sheaves $\mathcal{O}(\delta)$

Let $\delta = \Sigma \; v_i \zeta_{p_i}$ be a divisor on X. We get a sheaf $\mathcal{O}(\delta)$ as follows: If U is an open subset of X then $\Gamma(U, \mathcal{O}(\delta))$ consists of all meromorphic functions f on U satisfying $\mathrm{div}(f) \geqslant \delta$ in U. This means that f is holomorphic outside the points $\{ p_i \}$ and if $p_i \in U$ with $v_i > 0$ then we require that f has a zero of order $\geqslant v_i$ and if $v_i < 0$ we require that f has a pole of order $\leqslant v_i$.

In each chart U_α we can find some $f_\alpha \in \Gamma(U_\alpha, \mathcal{M}^*)$ such that $\operatorname{div}(f_\alpha) = \delta$ in U_α and it follows that the functions $\xi_{\alpha\beta} = f_\beta/f_\alpha$ is a cocycle in $Z^1(\mathcal{U}, \mathcal{O}^*)$ and hence they represent a cohomology class ξ in $H^1(X, \mathcal{O}^*)$.

We claim that ξ depends on the divisor δ only. In fact, if $\{ g_\alpha \}$ is another set of meromorphic functions satisfying $\operatorname{div}(g_\alpha) = \delta$ in each U_α amd if $\xi'_{\alpha\beta} = g_\beta/g_\alpha$ then we observe that $\xi_\alpha = g_\alpha/f_\alpha \in \Gamma(U_\alpha, \mathcal{O}^*)$ and we get $\xi'_{\alpha\beta} = (\xi_\beta/\xi_\alpha)\xi_{\alpha\beta}$ which means that the cocycles $\{ \xi_{\alpha\beta} \}$ and $\{ \xi'_{\alpha\beta} \}$ are cohomologous.

Summing up, a divisor δ gives a cohomology class in $H^1(X, \mathcal{O}^*)$ which we denote by $[\delta]$.

If $\xi = [\delta]$ then we get the sheaf $\mathcal{O}(\xi)$ and we claim

Lemma 2.3.4. $\mathcal{O}(\delta) \cong \mathcal{O}(\xi)$ as sheaves on X

Proof Let $\xi_{\alpha\beta} = \{ f_\beta / f_\alpha \}$ be a cocycle which represents $\xi = [\delta]$. The definition of the sheaf $\mathcal{O}(\delta)$ shows that if $f \in \Gamma(U, \mathcal{O}(\delta))$ then $\operatorname{div}(f) \geqslant \operatorname{div}(f_\alpha)$ in $U \cap U_\alpha$ and hence $g_\alpha = f/f_\alpha \in \Gamma(U \cap U_\alpha, \mathcal{O})$ and the family $\{ g_\alpha \}$ satisfies $g_\beta = (f_\alpha/f_\beta)g_\alpha = \xi_{\beta\alpha}g_\alpha$. These equations means that $\{ g_\alpha \} \in \Gamma(U, \mathcal{O}(\xi))$ and Lemma 2.3.4. follows.

2.3.5. The Riemann-Roch Theorem

Let δ be a divisor. Theorem 2.1. and Lemma 2.3.4. imply that $H^0(X, \mathcal{O}(\delta))$ and $H^1(X, \mathcal{O}(\delta))$ are finite dimensional complex vector spaces. Hence we can compute their dimensions and to simplify the notations we put: $h_0(\delta) = \dim_C(H^0(X, \mathcal{O}(\delta))$ and $h_1(\delta) = \dim_C(H^1(X, \mathcal{O}(\delta))$. Recall also that $|\delta|$ is the degree of the divisor. With these notations we can prove

Proposition 2.3.5. $h_0(\delta) - h_1(\delta) - |\delta| =$ a constant K for all divisors δ.

Proof If δ is a given divisor and if p is a point on X it is sufficient to prove that $h_0(\delta) - h_1(\delta) - |\delta| = h_0(\delta') - h_1(\delta') - |\delta'|$ where $\delta' = \delta + \zeta_p$. In fact, if this is proved we can move from the divisor δ to any other divisor in a finite number of steps, where we add or subtract single points on X.

So let $\delta' = \delta + \zeta_p$. This means that in the sheaf $\mathcal{O}(\delta')$ we allow sections to have a pole at p with one order > then the order of the pole at p for sections in $\mathcal{O}(\delta)$. Or, if δ contains p with some $v(p) > 0$ then sections in δ' are allowed to have zeros at p with multiplicity $v(p)-1$.

Outside the single point p, the two sheaves $\mathcal{O}(\delta)$ and $\mathcal{O}(\delta')$ are equal. Hence the quotient sheaf $\mathcal{O}(\delta')/\mathcal{O}(\delta)$ has stalks zero outside p and at the point p, the stalk is C. This means that $\mathcal{R} = \mathcal{O}(\delta')/\mathcal{O}(\delta)$ is a so called scyscraper sheaf. Obviously $\mathcal{R}$ is fine and $\Gamma(X, \mathcal{R}) = C$.

The exact sequence of sheaves $0 \to \mathcal{O}(\delta) \to \mathcal{O}(\delta') \to \mathcal{R} \to 0$ gives the long exact sequence $0 \to H^0(X, \mathcal{O}(\delta)) \to H^0(X, \mathcal{O}(\delta') \to \Gamma(X, \mathcal{R}) \to$ $\to H^1(X, \mathcal{O}(\delta)) \to H^1(X, \mathcal{O}(\delta')) \to 0$ and computing the dimensions of these finite dimensional complex vector spaces we see that

$h_o(\delta) - h_1(\delta) + \dim_C(\Gamma(X,\mathcal{R})) = h_o(\delta') - h_1(\delta')$

Finally, we observe that $|\delta'| = |\delta| + 1$ and since $\dim_C(\Gamma(X,\mathcal{R})) = 1$ the result follows.

Now we compute the constant which appears on Proposition 2.3.5. If $\delta = \delta_o$ is the zero divisor, i.e. no points with $v \neq 0$ appear, then $\mathcal{O}(\delta) = \mathcal{O}$ and $|\delta| = 0$ and we get

$$K = \dim(H^o(X,\mathcal{O})) - \dim(H^1(X,\mathcal{O}))$$

Since global holomorphic functions on X reduce to constants we see that $\dim(H^o(X,\mathcal{O})) = 1$. If we apply the Duality in Theorem 2.2. with $\xi = 1$ we get $(H^1(X,\mathcal{O})^* \cong H^o(X,\mathcal{O}(\mathcal{K})) = H^o(X,\mathcal{O}^{1,o})$ and hence $\dim(H^1(X,\mathcal{O})) = \dim(H^o(X,\mathcal{O}^{1,o}))$ and this integer is denoted by g and we conclude that $K = 1 - g$.

2.3.6. The integer g is called the genus of the Riemann surface X. Hence $g = \dim_C(\mathcal{A})$ where $\mathcal{A} = \Gamma(X,\mathcal{O}^{1,o})$ is the linear space of abelian differentials on X. It turns out that g is related to the topology on X. In fact, consider the sheaf C whose stalks are the complex field C. Then we can prove

Proposition 2.3.7. $\dim_C(H^1(X, \underline{C}) = 2g$

Proof If $f \in \Gamma(U,\mathcal{O})$ for some open set U then $df = \partial f \in \Gamma(U,\mathcal{O}^{1,o})$ and of course $f = 0$ if and only if f is locally constant, i.e. if $f \in \Gamma(U, \underline{C})$. In a chart U_α we can solve the equations $f_\alpha = \partial g/\partial z_\alpha$ which gives $f_\alpha dz_\alpha = \partial g$ and hence we get the exact sequence of sheaves

$0 \to \underset{\sim}{C} \to \mathcal{O} \xrightarrow{\partial} \mathcal{O}^{1,0} \to 0$ and the long exact sequence

$$0 \to H^0(X,\underset{\sim}{C}) \rightarrowtail H^0(X,\mathcal{O}) \to H^0(X,\mathcal{O}^{1,0}) \to H^1(X,\underset{\sim}{C}) \to H^1(X,\mathcal{O}) \to$$

$$\to H^1(X,\mathcal{O}^{1,0}) \twoheadrightarrow H^2(X,\underset{\sim}{C}) \to H^2(X,\mathcal{O})$$

Here we recall that $H^2(X,\mathcal{O}) = 0$ and that the Duality applied to $\xi = \mathcal{K}$ gives $\dim_C(H^1(\mathcal{O}^{1,0})) = \dim_C(H^0(X,\mathcal{O})) = 1$.

Of course $H^0(X, \underset{\sim}{C}) = C$ because X is connected . Finally, since X is oriented by the complex analytic charts it follows that $H^2(X, \underset{\sim}{C}) = C$ and the exact sequence above is reduced to the exact sequence

$$0 \to H^0(X,\mathcal{O}^{1,0}) \to H^1(X,\underset{\sim}{C}) \to H^1(X,\mathcal{O}) \to 0$$

Another application of Theorem 2.2. gives $\dim_C(H^1(X,\mathcal{O})) = \dim_C(H^0(X,\mathcal{O}^{1,0})) = g$ and we can read off Proposition 2.3.7.

2.3.8. <u>Remark</u> Using the De Rham isomorphism we deduce an interesting consequence of Proposition 2.3.7. In fact, $H^1(X, \underset{\sim}{C})$ is the quotient space $\Gamma(X,\mathcal{E}^1)/ d(\Gamma(X,\mathcal{E}))$ = closed 1-forms/ exact 1-forms

Now we observe that if $w \in \mathcal{A}$ then w is a closed differential 1-form and hence w has a cohomology class $[w]$ in $H^1(X,\underset{\sim}{C})$. The exact sequence which appears in the proof of Proposition 2.3.7. shows that the map from $\mathcal{A}$ into $H^1(X, \underset{\sim}{C})$ is injective, i.e. abelian differentials cannot be exact . This is easy to prove directly, for if we assume that $w = df$ for some $f \in C^{\infty}(X)$ and restrict the attention to a chart U_α , then $w = h_\alpha dz_\alpha = \partial f/\partial z_\alpha \, dz_\alpha + \partial f/\partial \bar{z}_\alpha \, d\bar{z}_\alpha$ and we get $\partial f/\partial \bar{z}_\alpha = 0$ which means that $f \in \Gamma(X,\mathcal{O})$ and hence f is a constant so that $df = 0$.

Summing up, $\mathcal{A}$ appears as a g-dimensional subspace of $H^1(X, \mathbb{C})$.

Now we can also consider the antiholomorphic differential 1-forms, i.e. differential 1-forms which in a chart U_α are given by $\varphi_\alpha d\bar{z}_\alpha$, where the complex conjugate functions $\varphi_\alpha \in \Gamma(U_\alpha, \mathcal{O})$.

If $\bar{\mathcal{A}}$ is the set of antiholomorphic differential 1-forms on X it is easily seen that $\mathcal{A} \cong \bar{\mathcal{A}}$ and hence they have equal dimensions.

Now $\mathcal{A} \oplus \bar{\mathcal{A}} \cong H^1(X, \underline{C})$ (must) holds (because dimensions are equal). So this means that if ω is some closed differential 1-form on the manifold X, then there exists some abelian differential φ and some $\varphi' \in \bar{\mathcal{A}}$ such that $w - (\varphi + \varphi')$ is exact.

2.3.9. The existence of meromorphic functions

If δ is a divisor, then $h_1(\delta)$ is non-negative and Proposition 2.3.5. implies that $h_o(\delta) \geqslant 1 - g - |\delta|$

So if $p \in X$ is a given point and if we choose $\delta = -(g+1)\zeta_p$, then $|\delta| = -g-1$ and we get $h_o(\delta) \geqslant 2$ which means that $\Gamma(X, \mathcal{O}(\delta))$ contains non-constant sections, i.e. there exists a non-constant meromorphic function f on X whose divisor satisfies $\operatorname{div}(f) \geqslant -(g+1)\zeta_p$, i.e. f is holomorphic outside p and the order of its pole at p is at most $-(g+1)$.

3. The Divisor Class group

In Section 2.3. we defined the sheaf $\mathcal{O}(\delta)$ when δ is a divisor and proved the existence of a unique cohomology class $\xi \in H^1(X,\mathcal{O}^*)$ such that $\xi = [\delta]$, i.e. $\mathcal{O}(\xi) \cong \mathcal{O}(\delta)$. Let us now prove the converse.

Proposition 3.1. If $\xi \in H^1(X,\mathcal{O}^*)$ then there exists a divisor δ on X such that $\xi = [\delta]$

Before we enter the proof we recall that ξ arises from some cocycle $\{\xi_{\alpha\beta}\} \in Z^1(\mathcal{U},X)$ and we can consider meromorphic sections which arise from a family $\{ f_\alpha \in \Gamma(U_\alpha,\mathcal{M}) : f_\alpha = \xi_{\alpha\beta} f_\beta \text{ in } U_\alpha \cap U_\beta \}$. We denote these globally defined meromorphic sections by $\Gamma(X,\mathcal{M}(\xi))$ and prove

Lemma 3.2. $\Gamma(X,\mathcal{M}(\xi)) \neq 0$

Proof We can use similar methods as in 2.3.9. In fact, fix some point $p \in X$ and if v is a positive integer we consider the sheaf $\mathcal{O}(\xi\, \zeta_p^{-v})$ whose sections in an open set U in X consists of families

$$\{ g_\alpha \in \Gamma(U_\alpha \cap U, \mathcal{M}) : g_\alpha = \xi_{\alpha\beta} g_\beta \text{ in } U \cap U_\alpha \cap U_\beta \quad \text{and } \operatorname{div}(g_\alpha) \geq -v\zeta_p ,$$

i.e. we require that g_α is holomorphic outside p and allow poles of order $\leq v$ at the point p $\}$. $\mathcal{O}(\xi\, \zeta_p^{-v})$ contains $\mathcal{O}(\xi)$ as a subsheaf and the quotient sheaf $\mathcal{R}_v = \mathcal{O}(\xi\, \zeta_p^{-v}) / \mathcal{O}(\xi)$ is a scyscarper sheaf whose stalks vanish at all points except at p where $(\mathcal{R}_v)_p \cong C^p$.

The exact sequence $0 \to \mathcal{O}(\xi) \to \mathcal{O}(\xi\zeta_p^{-v}) \to \mathcal{R}_v \to 0$ gives a long exact sequence of cohomology groups. Here $\dim_C(\Gamma(X,\mathcal{R}_v)) = v$ while $H^1(X,\mathcal{R}_v) = 0$ and we get the equation

$$\dim_C(H^0(X, \mathcal{O}(\xi\zeta_p^{-v}))) - \dim_C(H^1(X, \mathcal{O}(\xi\zeta_p^{-v}))) = v + A$$

where $A = \dim_C(H^0(X, \mathcal{O}(\xi)) - \dim_C(H^1(X, \mathcal{O}(\xi))$ is a finite constant, using Theorem 2.1.

In particular $\dim_C(H^0(X, \mathcal{O}(\xi\zeta_p^{-v}))) \geqslant v + A$ and if we choose $v = -A + 1$ we get a non-zero section in $\mathcal{O}(\xi\zeta_p^{-v})$ and this section is a globally defined meromorphic section in the sheaf $\mathcal{M}(\xi)$.

Proof of Proposition 3.1. Let $\{ \xi_{\alpha\beta} \} \varepsilon Z^1(\mathcal{U}, \mathcal{O}^{*})$ reprsent ξ. Lemma 3.2. gives a family $\{ f_\alpha \varepsilon \Gamma(U_\alpha, \mathcal{M}^{*}) : f_\alpha = \xi_{\alpha\beta} f_\beta$ in $U_\alpha \cap U_\beta \}$

Since $\operatorname{div}(\xi_{\alpha\beta}) = 0$ we get $\operatorname{div}(f_\alpha) = \operatorname{div}(f_\beta)$ in $U_\alpha \cap U_\beta$ and hence ther exists a divisor δ on X such that $\delta = -\operatorname{div}(f_\alpha)$ in each U_α. Finally, repeating the construction in 2.3.1. we get $\mathcal{O}(\xi) \cong \mathcal{O}(\delta)$.

Summing up, Proposition 3.1. shows that every $\xi \varepsilon H^1(X, \mathcal{O}^{*})$ arises from a divisor. Let us introduce the following notations.

Let $\mathcal{D}$ denote the abelian group of divisors on X, i.e. the elements in $\mathcal{D}$ are finite sums $\Sigma v_i \zeta_{p_i}$ where $\{ p_i \}$ is a finite set of points on X and $\{ v_i \}$ are integers.

In Section 2.3.1. we defined a mapping $\delta \rightarrow [\delta]$ from $\mathcal{D}$ into $H^1(X, {}^{*})$ and Proposition 3.1. shows that this mapping is surjective. It remains to describe its kernel.

For this purpose we introduce the principal divisors which arise from globally defined meromorphic functions on X when we put $\delta = \operatorname{div}(f)$

Let $\mathcal{P}$ denote the set of all principal divisors. Since div(fg) = = div(f) + div(g) obviously holds when f and g $\varepsilon\ \Gamma(X, \mathcal{M}^{*})$, we see that $\mathcal{P}$ is a subgroup of $\mathcal{D}$.

The quotient group $\mathcal{D}/\mathcal{P}$ is called the Divisor Class group on X.

With these notations we prove

Proposition 3.3. $\mathcal{D}/\mathcal{P} \cong H^1(X, \mathcal{O}^{*})$

Proof Let us first prove that $\mathcal{P} \subset$ in the kernel of the mapping $\delta \to [\delta]$ from $\mathcal{D}$ into $H^1(X, \mathcal{O}^{*})$. So let $\delta = \mathrm{div}(f)$ where $f\ \varepsilon\ \Gamma(X, \mathcal{M}^{*})$. Following the construction in 2.3.1. we can put $f_\alpha = f$ in each chart U_α and get the cocycle $\xi_{\alpha\beta} = f_\beta/f_\alpha = 1$ for all α and β. This means that $[\mathrm{div}(f)]$ is represented by the unit cocycle $\{ \xi_{\alpha\beta} \equiv 1 \}$ and the corresponding cohomology class is the unit element in $H^1(X, \mathcal{O}^{*})$ whose associated sheaf is $\mathcal{O}(1)$

Conversely, let $\delta\ \varepsilon\ \mathcal{D}$ and suppose that $[\delta] = 0$ in $H^1(X, \mathcal{O}^{*})$. This means that if we choose $\{ f_\alpha\ \varepsilon\ \Gamma(U_\alpha, \mathcal{M}^{*}) : \mathrm{div}(f_\alpha) = \delta \text{ in } U_\alpha \}$ then the cocycle $\{ \xi_{\alpha\beta} = f_\beta/f_\alpha \}\ \varepsilon\ B^1(\mathcal{U}, \mathcal{O}^{*})$. Hence there exist $\{ g_\alpha\ \varepsilon\ \Gamma(U_\alpha, \mathcal{O}^{*}) \}$ such that $g_\alpha/g_\beta = f_\beta/f_\alpha$ in $U_\alpha \cap U_\beta$ which gives $g_\alpha f_\alpha = g_\beta f_\beta$. This means that there exists $F\ \varepsilon\ \Gamma(X, \mathcal{M}^{*})$ such that $F = g_\alpha f_\alpha$ on U_α and $\mathrm{div}(F) = \mathrm{div}(g_\alpha) + \mathrm{div}(f_\alpha) = \mathrm{div}(f_\alpha) = \delta$ in each U_α. This means that $\delta = \mathrm{div}(F)\ \varepsilon\ \mathcal{P}$ and Proposition 3.3. is proved.

3.4. <u>The subgroup</u> $\mathcal{D}_o$. Recall that if $\delta \in \mathcal{D}$ then we get the integer $|\delta| = \Sigma v_i$ when $\delta = \Sigma v_i \zeta_{p_i}$ and we can consider the subgroup $\mathcal{D}_o = \{ \delta \in \mathcal{D} : |\delta| = 0 \}$. We prove

<u>Lemma</u> 3.4. $\mathcal{P} < \mathcal{D}_o$, i.e. $|\mathrm{div}(F)| = 0$ for all $F \in \Gamma(X, \mathcal{M}^*)$

<u>Proof</u> A meromorphic function F on X can be considered as a holomorphic mapping from the complex analytic manifold X into the Riemann sphere S^2. S^2 is the complex plane C^1 with the point at infinity added and $1/z$ is the local coordinate function at ∞. If $p \in X$ is a pole of F, then $F(p) = \infty$.

Put $\Omega = \{ p \in S^2 : p$ is a non-critical value of the mapping F, i.e. if $x \in F^{-1}(p)$ then F maps some open neighborhood of x biholomorphically onto an open neighborhood of p $\}$. It is easily seen that $S^2 \setminus \Omega$ is a finite set. In particular Ω is an open and connected subset of S^2 and if $p \in \Omega$ we get the integer $\# F^{-1}(p)$ = the number of points $x \in X$ satisfying $F(x) = p$. The function $p \to \# F^{-1}(p)$ is locally constant on Ω because F is biholomorphic in a neighborhood of each point in $F^{-1}(p)$. Since Ω is connected we get an integer $w = \# F^{-1}(p)$ for all $p \in \Omega$.

Finally, if $p \in S^2 \setminus \Omega$ we see that $\# F^{-1}(p) = w$ if we count the " p-values with their multiplicities ". We apply this when $p = \infty$ and when p is the origin 0 in $C^1 \subset S^2$ and get $|\mathrm{div}(F)| = w - w = 0$.

3.5. <u>Picard's variety</u> $\mathcal{J}(X)$

Using Lemma 3.4. we get the group $\mathcal{D}_o/\mathcal{P} = H_o^1(X, \mathcal{O}^*) = \{ \xi = | \delta | :$ $|\delta| = 0 \}$

Now we begin to represent this group in another way. Consider the exact sequence $0 \to \underline{Z} \to \mathcal{O} \xrightarrow{2\pi i \exp} \mathcal{O}^{*} \to 0$ which gives the exact séquence

$$0 \to H^1(X, \underline{Z}) \to H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \xrightarrow{c} H^2(X, \underline{Z}) \to 0$$

as proved in Proposition 1.5.2.

If $\xi = [\delta]$ is an element in $H^1(X, \mathcal{O}^*)$ we get the integer $|\delta|$. This integer also arises in the exact sequence above, using the topological fact that $H^2(X, \underline{Z})$ is the group of integers and hence the mapping

$c: H^1(X, \mathcal{O}^*) \to H^2(X, \underline{Z})$ assigns an integer $c(\xi)$ and it turns out that $c([\delta]) = |\delta|$ for all divisors δ.

In particular this proves that $\xi \in H_o^1(X, \mathcal{O}^*)$ if and only ξ arises from $H^1(X, \mathcal{O})$, which means that there exists an additive cocycle

$\{g_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O})$ such that $\{\xi_{\alpha\beta} = e^{2\pi i g_{\alpha\beta}}\}$ represents the cohomology class ξ.

Summing up, we have

<u>Lemma</u> 3.6. $\xi \in H_o^1(X, \mathcal{O}^*)$ if and only if ξ is represented by a cocycle of the form $\{e^{2\pi i g_{\alpha\beta}}\}$ where $\{g_{\alpha\beta}\} \in Z^1(\mathcal{U}, \mathcal{O})$

<u>Remark</u> Of course, the arguments above do not provide a detailed proof of Lemma 3.6. In particular we have used the topological fact which says that we always have $H^n(Y, \underline{Z}) \cong Z$ when Y is an orientable compact manifold of dimension n. This is applied with $Y = X$ and $n = 2$ here.

For the sake of completeness we give a direct proof of one half of Lemma 3.6. Namely, we can prove

<u>Lemma</u> 3.7. Let $\delta \in \mathcal{D}_o$. Then $[\delta] \in$ the image of $H^1(X, \mathcal{O})$.

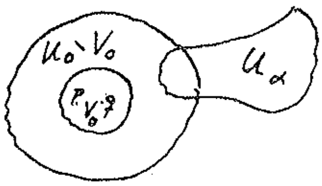
<u>Proof</u> The image of $H^1(X, \mathcal{O})$ is a subgroup of $H^1(X, \mathcal{O}^*)$ and if $|\delta| = 0$ we can write δ as a finite sum $\Sigma\, \delta_v$, where $\delta_v = \zeta_{q_v} - \zeta_{p_v}$ for some pair

of points p_ν and q_ν in X. Hence it is sufficient to prove Lemma 3.7. for each δ_ν.

So let p and q be two given points on X. Then we can choose a finite string of points $q = q_o, q_1, \ldots\ q_s = p$ where each pair $(q_\nu, q_{\nu+1})$ belongs to a chart in X.

Since $\zeta_q - \zeta_p = (\zeta_{q_o} - \zeta_{q_1}) + \ldots + (\zeta_{q_s} - \zeta_{q_{s-1}})$, it remains only to prove Lemma 3.7. when $\delta = \zeta_q - \zeta_p$, where q and p are so close to each other that there exists a chart U_o in X containing both q and p.

This means that we can choose a Leray covering $\{ U_\alpha \}$, where U_o appears and in addition this can be arranged in such a way that U_o contains a relatively compact open subset V_o, where both p and q ε V_o while $U_\alpha \cap \overline{V}_o$ are empty for all $\alpha \neq 0$. This is illustrated by the figure below



In particular we can assume that U_o is biholomorphic to an open disc and V_o is a smaller disc inside U_o, so that $U_o \setminus \overline{V}_o$ is an open annulus.

Let us now choose a meromorphic function f_o on U_o such that $\mathrm{div}(f_o) = \zeta_q - \zeta_p$. Hence f has a simple zero and a simple pole in V_o and f is holomorphic and $\neq 0$ in the annulus $U_o \setminus \overline{V}_o$. It follows that we can select a single valued branch of log f_o in this annulus and we put

$$g_o = (2\pi i)^{-1} \log f_o \quad \text{in} \quad U_o \setminus \overline{V}_o .$$

If $\alpha \neq 0$ then $U_\alpha \cap U_o \subset U_o \setminus \overline{V}_o$ and hence we can introduce the holomorphic function $g_{\alpha o} = g_o$ in $U_\alpha \cap U_o$ and put $g_{o\alpha} = -g_{\alpha o}$. If both α and β are $\neq 0$ we put $g_{\alpha\beta} = 0$. Then

$\{ g_{\alpha\beta} \}$ is an alternating 1-cochain of $\mathcal{U}$ with values in $\mathcal{O}$ and the construction of $[\delta]$ in Section 2.3.1. shows that

$[\delta]$ is represented by $\{ e^{2\pi i\, g_{\alpha\beta}} \}$

Summing up, we have now established the following 3 isomorphisms.

<u>Lemma</u> 3.8. $H_o^1(X,\mathcal{O}^*) \cong \mathcal{D}_o/\mathcal{P} \cong \dfrac{H^1(X,\mathcal{O})}{H^1(X,\underline{Z})}$

At this stage we consider the exact sequence

$0 \to \underline{C} \longrightarrow \mathcal{O} \xrightarrow{\partial} \mathcal{O}^{1,0} \longrightarrow 0$ which gives the long exact sequence

$0 \to H^0(X,\mathcal{O}^{1,0}) \to H^1(X,\underline{C}) \to H^1(X,\mathcal{O}) \longrightarrow 0$, as observed just above

2.3.8. Remark.

Recall that $H^1(X,\underline{C})$ is a 2g-dimensional complex vector space and identifying $H^1(X,\underline{Z})$ with a subgroup of $H^1(X,\underline{C})$, we see that the quotient group $H^1(X,\mathcal{O})/H^1(X,\underline{Z})$ is isomorphic to the quotient group

$$\mathcal{J}(X) = \frac{H^1(X,C)}{H^1(X,\underline{Z}) + H^0(X,\mathcal{O}^{1,0})}$$

$\mathcal{J}(X)$ is called the <u>Picard group</u> of X and we shall soon begin to describe this group in more detail. Let us first observe that Lemma 3.8. now can be rewritten as

<u>Proposition</u> 3.9. $H_o^1(X,\mathcal{O}^*) \cong \mathcal{D}_o/\mathcal{P} \cong \mathcal{J}(X)$

We begin the study of $\mathcal{J}(X)$. Let us first recall

3.10. <u>The cup product on</u> $H^1(X,\underline{C})$.

First, De Rham's Theorem shows that $H^1(X,\underline{C})$ = closed/exact differential 1-forms on X.

Consider next two 1-forms w and η on X. The exterior product $w\wedge\eta$ is a 2-form which can be integrated on the compact and orientable manifold x and gives the complex scalar $\int_X w\wedge\eta$

Since $\cup$ is (a) non-degnerated bilinear form on $H^1(X, \underline{C})$, we can idenŝtify $H^1(X, \underline{C})$ with its own dual space.

In particular, let Γ be some differentiable curve on X. That is, Γ arises from a differentiable mapping γ: $[0,] \to X$. Then we can integrate differential 1-forms along Γ and the map which sends

$$\sigma = \overline{w} \longrightarrow \int_\Gamma w$$ is a linear form on $H^1(X, \underline{C})$. Hence there exists a unique element $\sigma(\Gamma)$ such that

$$\sigma(\Gamma) \cup \sigma = \int_\Gamma w \quad \text{for all } \sigma = \overline{w} \text{ in } H^1(X, \underline{C})$$

Of course, we can consider differentiable curves which arise as finite sums of curves arising from maps $\gamma: [0,1] \to X$ and in particular we can consider <u>loops</u> which consist of <u>finite sums of closed curves</u>.

Since we restrict the attention to closed 1-forms when we compute $\sigma(\Gamma)$ for a given curve Γ, we see that $\sigma(\Gamma)$ depends on the homotopy class of Γ only.

At this stage we understand the position of $H^1(X, \underline{Z})$ in $H^1(X, \underline{C})$. In fact, $H^1(X, \underline{Z}) \cong H_1(X, \mathbb{Z})^*$ and the singular homology with values in Z is a free abelian group of rank 2g, where a free basis is provided by 2g simple closed curves $\Gamma_1 \dots \Gamma_{2g}$, and the homology class of an arbitrary closed curve $\Gamma = \sum_{i=1}^{i=2g} v_i \Gamma_i$ for some integers $v_1 \dots v_{2g}$.

Summing up, the whole discussion above gives

<u>Proposition</u> 3.12. Let $\sigma \in H^1(X, \underline{C})$. Then σ belongs to the subgroup $H^1(X, \underline{Z})$ if and only if there exists a differentiable loop $\mathcal{T}$ on X such that $\sigma \cup \overline{w} = \int_{\mathcal{T}} w$ for all closed 1-forms w on X.

Recall that $\mathcal{J}(X) = H^1(X,\underline{C}) / (H^1(X,\underline{Z}) + \mathcal{A})$. The position of $H^1(X,\underline{Z}) + \mathcal{A}$ inside $H^1(X, \underline{C})$ is clarified by the next result.

<u>Proposition</u> 3.13. Let $\sigma \in H^1(X, \underline{C})$ be given. Then $\sigma \in H^1(X,\underline{Z}) + \mathcal{A}$ if and only if there exists a differentiable loop γ in X such that

$$\sigma \cup \varphi = \int_\gamma \varphi \quad \text{for all } \varphi \in \mathcal{A}$$

<u>Proof</u> Suppose first that $\sigma = \sigma(\gamma) + \varphi_o$, where γ is a loop and $\varphi_o \in \mathcal{A}$

Since $\varphi \cup \varphi_o = 0$ for all $\varphi \in \mathcal{A}$ we get

$$\sigma \cup \varphi = \int_\gamma \varphi \quad \text{for all } \varphi \in \mathcal{A} .$$

Conversely, assume that there exists a loop γ such that $\sigma \cup \varphi = \int_\gamma \varphi$ for all $\varphi \in \mathcal{A}$.

Let $\varphi_1 \dots \varphi_g$ be a basis for the complex vector space $\mathcal{A}$. The proof of Lemma 3.11. shows that the matrix $(\varphi_j \cup \overline{\varphi}_v)_{j,v=1}^{j,v=g}$ is non-singular.

Introduce the complex scalars $d_v = \int_\gamma \overline{\varphi}_v - \sigma \cup \overline{\varphi}_v$

and solve the system of equations

$$\sum_{j=1}^{j=g} c_j (\varphi_j \cup \overline{\varphi}_v) = d_v \quad \text{for } 1 < v < g.$$

Using these g complex scalars $c_1 \dots c_g$ we put $\sigma' = \sigma + c_1\varphi_1 + \dots + c_g\varphi_g$ and then $\sigma' \cup \overline{\varphi}_v = \int_\gamma \overline{\varphi}_v$ for each $1 \leqslant v \leqslant g$ and $\sigma' \cup \varphi_j = \sigma \cup \varphi_j = \int_\gamma \varphi_j$ for all $1 \leqslant j \leqslant g$. It follows from Proposition 3.12. that $\sigma' \in H^1(X, \underline{Z})$ and we conclude that $\sigma \in H^1(X, \underline{Z}) + \mathcal{A}$, as required.

3.14. <u>The mapping from $\mathcal{D}_o/\wp$ into $\mathcal{J}(X)$</u>

Let us first consider a divisor $\delta = \zeta_q - \zeta_p$ where q and p are two points on X. Then δ has an image in $\mathcal{D}_o/\wp \cong \mathcal{J}(X)$ and we let $j(\delta)$ denote the corresponding element in $\mathcal{J}(X)$.

Proposition 3.13. shows that if $j \in \mathcal{J}(X)$ is given and if σ and σ' are two elements in $H^1(X, \underline{C})$ whose images both give j, then there exists a loop τ on X such that $\sigma \cup \varphi - \sigma' \cup \varphi = \int_\tau \varphi$ for all $\varphi \in \mathcal{A}$.

This can be expressed by saying that $\mathcal{J}(X)$ is identified with the dual space of $\mathcal{A}$/ those linear forms which arise when we integrate abelian differentials along loops in X.

After this digression we announce

<u>Proposition</u> 3.14. Let $\delta = \zeta_q - \zeta_p$ and choose some differentiable curve Γ in X which joins q to p. Then $j(\delta)$ is the element in $\mathcal{J}(X)$ which arises from the linear form $\varphi \to \int_\Gamma \varphi$ on $\mathcal{A}$.

<u>Remark</u> Observe that the resulting element $j(\delta)$ depends on the two points q and p only. For if Γ' is another curve which joins q to p, then $(\Gamma') \circ \Gamma^{-1}$ is a closed curve ,and so on.

<u>Proof of Proposition</u> 3.14. By the same method as in the beginning of the proof of Lemma 3.7. we can assume that q and p belong to a chart U_o where we have $p,q \in V_o \subset\subset U_o$ and a Leray covering $\{ U_\alpha \}$ where U_o appears and $U_\alpha \cap \overline{V_o}$ are empty for all $\alpha \neq 0$.

During the proof of Lemma 3.7. we constructed the cocycle $\{ g_{\alpha\beta} \}$ in $Z^1(\mathcal{U}, \mathcal{O})$ where we recall that $g_{\alpha o} = - g_{o\alpha} = (2\pi i)^{-1} \log f_o$ and f_o is a meromorphic function in U_o with $\operatorname{div}(f_o) = -\zeta_q + \zeta_p$.

Notice the rule for signs which arise in the construction in 153

Now $\{ g_{\alpha\beta} \} \varepsilon H^1(X,\mathcal{O})$ and in the exact sequence

$0 \longrightarrow H^1(X,\mathcal{O}^{1,0}) \longrightarrow H^1(X,\underline{C}) \longrightarrow H^1(X,\mathcal{O}) \longrightarrow 0$ we get an element $\sigma \varepsilon H^1(X, \underline{C})$ whose image in $H^1(X, \mathcal{O})$ is the cohomology class represented by the cocycle $\{ g_{\alpha\beta} \}$.

Identifying $H^1(X,\underline{C})$ with closed/exact 1-forms, and tracing through the identifications, it follows that σ corresponds to a closed 1-form wihch can be constructed as follows:

Let $\{ h_\alpha \}$ be a partition of the unity, subordinate to the covering $\{ U_\alpha \}$. That is, each $h_\alpha \varepsilon C_0^\infty(U_\alpha)$ and $\Sigma h_\alpha = 1$. Since $V_0 \subset U_0$ and $U_\alpha \cap \overline{V}_0$ are empty when $\alpha \neq 0$, it follows that h_0 satisfies $h_0 = 1$ in a neighborhood of $\overline{V}_0$.

To each α we put $H_\alpha = \Sigma h_\beta g_{\alpha\beta}$. Then $\{ H_\alpha \varepsilon C^\infty(U_\alpha) \}$ and since $\{ g_{\alpha\beta} \} \varepsilon Z^1(\mathcal{U},\mathcal{O})$ and $\sum_\alpha h_\alpha = 1$, a compuation shows that the (o,1)-forms $\bar\partial H_\alpha = \bar\partial H_\beta$ in U_α U_β. Hence there exists a glo ally defined (o,1) -form ω on X satisfying $\omega = \bar\partial H_\alpha$ in U_α.

ω is a closed form and all our identifications give

$j(\delta)$ = the image of ω in $\mathcal{J}(X)$. Hence Proposition 3.14. follows if we can prove that $\overline{\omega} \cup \varphi = \int_X \omega \wedge \varphi = \int_\Gamma \varphi$ for all $\varphi \varepsilon \mathcal{A}$, where Γ of course can be chosen so that it joints q and p and stays inside the open disc V_0.

To prove the equality above we recall that $g_{\alpha\beta} = 0$ when both α and β are $\neq 0$ while $g_{\alpha 0} = - g_{0\alpha} = (2\pi i)^{-1} \log f_0 = g_0$

We conclude that if $\alpha \neq 0$ then

$$H_\alpha = \Sigma h_\beta g_{\alpha\beta} = h_0 g_{\alpha 0} = h_0 g_0 \quad \text{and } H_0 = \Sigma h_\beta g_{0\beta} = -(1-h_0) g_0 =$$

$= -g_o + h_o g_o$ and it follows that $\omega = g_o \bar{\partial} h_o$.

In particular $\mathrm{supp}(\omega) \subset \mathrm{supp}(\bar{\partial} h_o) \subset\subset U_o \setminus \bar{\bar{V}}_o$

Let us also recall that $h_o = 1$ in a neighborhood of $\bar{\bar{V}}_o$.

To make everytihng explicit we can assume that we are working in a disc in the complex z-plane (corresponing to the chart U_o) and assume that

$h_o(z) = 1$ when $|z| = 1$ while $\mathrm{supp}(\bar{\partial} h_o) \subset 1 < |z| < 2$

and finally g_o is holomorphic and $= (2\pi i)^{-1} \log f_o$ in the annulus

$1/2 < |z| < 2$ and q and p belong to $|z| < 1/2$.

Finally, if $\varphi \in \mathcal{A}$ is given we can write $\varphi = \varphi_o(z)dz$ in $|z|<2$ where φ_o is holomorphic

Let us now consider the differential form

$\varphi_o g_o h_o dz$ in the open annulus $1 < |z| < 2$

Since $\varphi_o g_o$ is holomorphic there we see that

$$d(\varphi_o g_o h_o \, dz) = \bar{\partial}(\varphi_o g_o \, h_o \, dz) = \omega \wedge \varphi$$

and Stokes Theorem gives

$$\int_X \omega \wedge \varphi = \int_{1<|z|<2} \omega \wedge \varphi = -\int_{|z|=1} \varphi_o g_o dz$$

where we used the fact that $\mathrm{supp}(\omega) \subset U_{oo} \setminus \bar{V}_o$ and that $h_o(z) = 1$ when $|z| = 1$ while $h_o(z) = 0$ when $|z| = 2$.

Now φ_o is holomorphic in the disc $|z| < 2$ so we can choose a primitive function Φ , i.e. Φ is holomorphic in $|z| < 2$ and $\Phi' = \varphi_o$

A partial integration gives $-\int_{|z|=1} \varphi_o g_o dz = \int_{|z|=1} \Phi g_o' \, dz$

Finally, $g_o' = (2\pi i)^{-1} f'_o / f_o$ and the Residue Theorem shows that

the last integral is $\Phi(q) - \Phi(p) = \int_\Gamma \varphi$. This completes the proof.