Fiscal Activism under Inflation Targeting and Non-atomistic Wage setting: Technical Appendix

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Derivations

Households

Household h solves:

$$\max_{C_{hij},M_h} U_h = \left(\frac{1}{\alpha}\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}}\,dj\right)^{\frac{\theta}{\theta-1}}\right)^{\alpha}\left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha} + \eta G - \frac{\beta}{2}G^2$$

subject to

$$\int_0^1 C_{hij}P_{ij}dj + M_h \equiv PC_h + M_h = X_h$$

Denote the nominal expenditure spent on consumption by Z_h:

$$Z_h \equiv X_h - M_h = PC_h$$

The household's problem may be solved in two steps. Given a fixed nominal amount spent on consumption, Z_h, the household first chooses how much to consume of each good C_{hij} as a function of total household consumption C_h. The household then chooses how to allocate total nominal income, X_h, between consumption C_h and money holdings M_h.

In the first step, the Lagrangian can be written: [1]

$$L\left(C_{hij},\widetilde{\xi}\right) = \left(\frac{1}{\alpha}\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}}\,dj\right)^{\frac{\theta}{\theta-1}}\right)^{\alpha}\left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha} + \eta G - \frac{\beta}{2}G^2 + \widetilde{\xi}\left[Z_h - \int_0^1 C_{hij}P_{ij}dj\right]$$

The first-order conditions are:

$$\alpha\left(\frac{1}{\alpha}\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}}\,dj\right)^{\frac{\theta}{\theta-1}}\right)^{\alpha-1}\left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha}\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}}\,dj\right)^{\frac{\theta}{\theta-1}-1} C_{hij}^{\frac{\theta-1}{\theta}-1} - \widetilde{\xi}P_{ij} = 0 \tag{1}$$

$$Z_h - \int_0^1 C_{hij}P_{ij}dj = 0 \tag{2}$$

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[1] Let ˜ denote intermediate parameters, i.e. parameters (such as multipliers) that are introduced for computational purposes only.

Simplifying the FOC with respect to C_{hij}:

$$\alpha C_h \left(\frac{1}{\alpha} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \right)^{-1} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} C_{hij}^{\frac{\theta-1}{\theta}} = \widetilde{\xi} C_{hij} P_{ij} \left(\frac{M_h/P}{1-\alpha} \right)^{-(1-\alpha)} \tag{3}$$

Integrating both sides over $(0, 1)$:

$$\alpha C_h \left(\frac{1}{\alpha} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \right)^{-1} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} \int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj$$
$$= \widetilde{\xi} \left(\frac{M_h/P}{1-\alpha} \right)^{-(1-\alpha)} \int_0^1 C_{hij} P_{ij} dj$$

if and only if

$$\alpha C_h \left(\frac{1}{\alpha} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \right)^{-1} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} = \widetilde{\xi} \left(\frac{M_h/P}{1-\alpha} \right)^{-(1-\alpha)} \int_0^1 C_{hij} P_{ij} dj \tag{4}$$

Imposing (2) on (4) and using the definition of C_h implies:

$$\alpha C_h \left(\frac{1}{\alpha} C_h \right)^{-1} C_h = \widetilde{\xi} \left(\frac{M_h/P}{1-\alpha} \right)^{-(1-\alpha)} Z_h \Leftrightarrow$$
$$\widetilde{\xi} = \frac{\alpha^2 C_h}{Z_h} \left(\frac{M_h/P}{1-\alpha} \right)^{(1-\alpha)}$$

Plugging this into the first-order condition (3) and using $Z_h = PC_h$:

$$\alpha C_h \left(\frac{1}{\alpha} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \right)^{-1} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} C_{hij}^{\frac{\theta-1}{\theta}} = \widetilde{\xi} C_{hij} P_{ij} \left(\frac{M_h/P}{1-\alpha} \right)^{-(1-\alpha)}$$
$$\alpha C_h \left(\frac{1}{\alpha} C_h \right)^{-1} \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} C_{hij}^{\frac{\theta-1}{\theta}} = \frac{\alpha^2 C_h}{Z_h} C_{hij} P_{ij} \left(\frac{M_h/P}{1-\alpha} \right)^{(1-\alpha)-(1-\alpha)} \Leftrightarrow$$
$$\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}-1} C_{hij}^{\frac{\theta-1}{\theta}} = \frac{P_{ij}}{P} C_{hij} \Leftrightarrow$$
$$\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{1}{\theta-1}} C_{hij}^{-\frac{1}{\theta}} = \frac{P_{ij}}{P}$$

Note that:

$$\left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{1}{\theta-1}} = \left(\int_0^1 C_{hij}^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1} \frac{\theta-1}{\theta} \frac{1}{\theta-1}} = C_h^{\frac{\theta-1}{\theta} \frac{1}{\theta-1}} = C_h^{\frac{1}{\theta}}$$

I then obtain:

$$\left(\frac{C_h}{C_{hij}} \right)^{\frac{1}{\theta}} = \frac{P_{ij}}{P} \Leftrightarrow$$
$$C_h = \left(\frac{P_{ij}}{P} \right)^{\theta} C_{hij}$$

Rearranging implies:

$$C_{hij} = \left(\frac{P_{ij}}{P}\right)^{-\theta} C_h$$

Instead of explicitly deriving money demand from the above system, I want to derive the trade-off between total consumption of household h, C_h, and money holdings $\frac{M_h}{P}$. Therefore, I aggregate over goods and consider the following (equivalent problem):

$$\max_{C_h, M_h} \left(\frac{C_h}{\alpha}\right)^{\alpha} \left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha} + \eta G - \frac{\beta}{2} G^2$$

subject to

$$PC_h + M_h = X_h.$$

The Lagrangian is:

$$L(C_h, M_h, \widetilde{\lambda}) = \left(\frac{C_h}{\alpha}\right)^{\alpha} \left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha} + \eta G - \frac{\beta}{2} G^2 + \widetilde{\lambda}\left[X_h - PC_h - M_h\right].$$

The first-order conditions are:

$$\begin{aligned}
\alpha \left(\frac{C_h}{\alpha}\right)^{\alpha-1} \frac{1}{\alpha} \left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha} - \widetilde{\lambda} P = 0 \\
-(1-\alpha) \left(\frac{C_h}{\alpha}\right)^{\alpha} \frac{1}{(1-\alpha)} \frac{1}{P} \left(\frac{M_h/P}{1-\alpha}\right)^{1-\alpha-1} - \widetilde{\lambda} = 0 \\
X_h - PC_h - M_h = 0.
\end{aligned}$$

Re-arranging means that the FOC with respect to C_h can be written:

$$\begin{aligned}
\left(\frac{\frac{M_h/P}{1-\alpha}}{\frac{C_h}{\alpha}}\right)^{1-\alpha} - \widetilde{\lambda} P = 0 \Leftrightarrow \\
\left(\frac{M_h/P}{C_h} \frac{\alpha}{1-\alpha}\right)^{1-\alpha} - \widetilde{\lambda} P = 0 \Leftrightarrow \\
P = \frac{1}{\widetilde{\lambda}} \left(\frac{M_h/P}{C_h} \frac{\alpha}{1-\alpha}\right)^{1-\alpha}.
\end{aligned}$$

Similarly, the FOC with respect to money holdings can be written:

$$\begin{aligned}
\frac{1}{P} \left(\frac{\frac{C_h}{\alpha}}{\frac{M_h/P}{1-\alpha}}\right)^{\alpha} - \widetilde{\lambda} &= 0 \\
\left(\frac{C_h}{M_h/P} \frac{1-\alpha}{\alpha}\right)^{\alpha} - \widetilde{\lambda} P &= 0 \\
\frac{1}{\widetilde{\lambda} P} \left(\frac{M_h/P}{C_h} \frac{\alpha}{1-\alpha}\right)^{-\alpha} &= 1
\end{aligned}$$

Inserting these expressions into the FOC for $\widetilde{\lambda}$:

$$PC_h + M_h = X_h \Leftrightarrow$$

$$\frac{1}{\widetilde{\lambda}}\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{1-\alpha} C_h + \frac{1}{\widetilde{\lambda}}\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P} = X_h \Leftrightarrow$$

$$\left(\frac{M_h/P}{C_h}\right)^{1-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} C_h + \left(\frac{M_h/P}{C_h}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P} = \widetilde{\lambda} X_h \Leftrightarrow$$

$$\left(\frac{M_h/P}{C_h}\right)\left(\frac{M_h/P}{C_h}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} C_h$$

$$+\left(\frac{M_h/P}{C_h}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P} = \widetilde{\lambda} X_h$$

If and only if:

$$\left(\frac{M_h/P}{C_h}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)\left(\frac{M_h/P}{C_h}\right) C_h + \frac{M_h}{P}\right] = \widetilde{\lambda} X_h \Leftrightarrow$$

$$\left(\frac{M_h/P}{C_h}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P}\left[\frac{\alpha}{1-\alpha}+1\right] = \widetilde{\lambda} X_h \Leftrightarrow$$

$$\left(\frac{M_h/P}{C_h}\right)^{-\alpha}\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P}\left[\frac{1}{1-\alpha}\right] = \widetilde{\lambda} X_h \Leftrightarrow$$

$$\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P}\left[\frac{1}{1-\alpha}\right]\frac{1}{X_h} = \widetilde{\lambda}.$$

Inserting this into the FOC for money holdings:

$$\begin{aligned}
\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha} &= \widetilde{\lambda} P \Rightarrow \\
\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha} &= \left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P}\left[\frac{1}{1-\alpha}\right]\frac{1}{X_h}P \Leftrightarrow \\
1 &= \frac{M_h}{P}\left[\frac{1}{1-\alpha}\right]\frac{1}{X_h}P \Leftrightarrow \\
\frac{M_h}{P} &= (1-\alpha)\frac{X_h}{P}
\end{aligned}$$

Similarly, the FOC for household consumption implies:

$$\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{1-\alpha} = \widetilde{\lambda} P \Leftrightarrow$$

$$\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)\left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha} = \left(\frac{M_h/P}{C_h}\frac{\alpha}{1-\alpha}\right)^{-\alpha}\frac{M_h}{P}\left[\frac{1}{1-\alpha}\right]\frac{1}{X_h}P \Leftrightarrow$$

$$\left(\frac{\alpha}{C_h}\right) = \frac{1}{X_h}P \Leftrightarrow$$

$$C_h = \alpha\frac{X_h}{P}.$$

Combining the two expressions gives the following relationship between household consumption and real money holdings:

$$C_h = \frac{\alpha}{(1-\alpha)}\frac{M_h}{P}.$$

This means that household $h's$ demand for goods provided by firm j can be written

$$\begin{aligned} C_{hij} &= \left(\frac{P_{ij}}{P}\right)^{-\theta} C_h \\ &= \left(\frac{P_{ij}}{P}\right)^{-\theta} \left[\frac{\alpha}{(1-\alpha)} \frac{M_h}{P}\right]. \end{aligned}$$

Finally, aggregate demand facing firm j, represented by union i, is obtained by integrating over all households on the unit interval:

$$\begin{aligned} Y_{ij}^D &\equiv \int_0^1 C_{hij} dh = \int_0^1 \left(\frac{P_{ij}}{P}\right)^{-\theta} \left[\frac{\alpha}{(1-\alpha)} \frac{M_h}{P}\right] dh \\ &= \frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta} \frac{M}{P}. \end{aligned}$$

Firms

Substituting for the constraints, the problem of the firm can be written:

$$\begin{aligned} \max_{P_{ij}} \Pi_{ij} &= \frac{P_{ij}}{P} Y_{ij}^D - \frac{W_i}{P} \left(Y_{ij}^D\right)^{\frac{1}{\gamma}} \\ &= \frac{P_{ij}}{P} \left[\frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta} \frac{M}{P}\right] - \frac{W_i}{P} \left(\frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta} \frac{M}{P}\right)^{\frac{1}{\gamma}}. \end{aligned}$$

Taking P as given. The FOC is:

$$\begin{aligned} \frac{\partial \Pi_{ij}}{\partial P_{ij}} = \frac{1}{P} \left[\frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta} \frac{M}{P}\right] - \theta \frac{P_{ij}}{P} \left[\frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta-1} \frac{M}{P}\right] \frac{1}{P} \\ -\frac{1}{\gamma} \frac{W_i}{P} \left(\frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta} \frac{M}{P}\right)^{\frac{1}{\gamma}-1} (-\theta) \frac{\alpha}{(1-\alpha)} \left(\frac{P_{ij}}{P}\right)^{-\theta-1} \frac{M}{P} \frac{1}{P} = 0. \end{aligned}$$

Simplifying implies:

$$\begin{aligned} \left(\frac{P_{ij}}{P}\right)^{-\theta} - \theta \left(\frac{P_{ij}}{P}\right)^{-\theta-1+1} &= -\frac{\theta}{\gamma} \frac{W_i}{P} \left(\frac{\alpha}{(1-\alpha)} \frac{M}{P}\right)^{\frac{1}{\gamma}-1} \left(\frac{P_{ij}}{P}\right)^{-\theta\left(\frac{1}{\gamma}-1\right)-\theta-1} \Leftrightarrow \\ (\theta-1) \left(\frac{P_{ij}}{P}\right)^{-\theta} &= \frac{\theta}{\gamma} \frac{W_i}{P} \left(\frac{\alpha}{(1-\alpha)} \frac{M}{P}\right)^{\frac{1}{\gamma}-1} \left(\frac{P_{ij}}{P}\right)^{-\frac{\theta}{\gamma}-1} \end{aligned}$$

if and only if

$$\begin{aligned} \left(\frac{P_{ij}}{P}\right)^{1-\theta+\frac{\theta}{\gamma}} &= -\frac{\theta}{\gamma(1-\theta)} \frac{W_i}{P} \left(\frac{\alpha}{(1-\alpha)} \frac{M}{P}\right)^{\frac{1-\gamma}{\gamma}} \Leftrightarrow \\ \left(\frac{P_{ij}}{P}\right)^{\frac{\gamma-\theta\gamma+\theta}{\gamma}} &= \frac{\theta}{\gamma(\theta-1)} \frac{W_i}{P} \left(\frac{\alpha}{(1-\alpha)} \frac{M}{P}\right)^{\frac{1-\gamma}{\gamma}}. \end{aligned}$$

This implies

$$
\begin{aligned}
\left(\frac{P_{ij}}{P}\right) &= \left[\frac{\theta}{\gamma\left(\theta-1\right)}\frac{W_i}{P}\left(\frac{\alpha}{\left(1-\alpha\right)}\frac{M}{P}\right)^{\frac{1-\gamma}{\gamma}}\right]^{\frac{\gamma}{\gamma+\theta(1-\gamma)}} \Leftrightarrow \\
\left(\frac{P_{ij}}{P}\right) &= \zeta\left(\frac{W_i}{P}\right)^{\frac{\gamma}{\gamma+\theta(1-\gamma)}}\left(\frac{M}{P}\right)^{\frac{(1-\gamma)}{\gamma+\theta(1-\gamma)}},
\end{aligned}
$$

where $\zeta \equiv \left[\frac{\theta}{\gamma(\theta-1)}\left(\frac{\alpha}{(1-\alpha)}\right)^{\frac{1-\gamma}{\gamma}}\right]^{\frac{\gamma}{\gamma+\theta(1-\gamma)}}$. Inserting the firm's optimal pricing rule into the expression for labour demand and simplifying, I obtain firm $ij's$ demand for labour:

$$
\begin{aligned}
L_{ij}^{P} &= \left[\frac{\alpha}{\left(1-\alpha\right)}\frac{M}{P}\right]^{\frac{1}{\gamma}}\left(\zeta\left(\frac{W_i}{P}\right)^{\frac{\gamma}{\gamma+\theta(1-\gamma)}}\left(\frac{M}{P}\right)^{\frac{(1-\gamma)}{\gamma+\theta(1-\gamma)}}\right)^{-\frac{\theta}{\gamma}} \\
&= \vartheta\left[\frac{M}{P}\right]^{\frac{1}{\gamma+\theta(1-\gamma)}}\left(\frac{W_i}{P}\right)^{-\frac{\theta}{\gamma+\theta(1-\gamma)}},
\end{aligned}
$$

where $\vartheta \equiv \left[\frac{\alpha}{1-\alpha}\right]^{\frac{1}{\gamma}}\zeta^{-\frac{\theta}{\gamma}} = \left[\frac{\alpha}{(1-\alpha)}\right]^{\frac{1}{\gamma+\theta(1-\gamma)}}\left[\frac{\theta}{\gamma(\theta-1)}\right]^{-\frac{\theta}{\gamma+\theta(1-\gamma)}}$. Using this notation, the price setting rule of each firm can be written:

$$
\left(\frac{P_{ij}}{P}\right) = \zeta\left(\frac{W_i}{P}\right)^{\gamma\phi}\left(\frac{M}{P}\right)^{(1-\gamma)\phi},
$$

where $\zeta \equiv \left[\frac{\alpha}{1-\alpha}\right]^{(1-\gamma)\phi}\left[\frac{\theta}{\gamma(\theta-1)}\right]^{\gamma\phi}$.

Aggregation

To obtain an expression for aggregate unemployment, I need to derive an expression for aggregate labour demand. Recall that labour demand facing firm ij is given by:

$$
L_{ij} = \left[\frac{\alpha}{(1-\alpha)}\left(\frac{P_{ij}}{P}\right)^{-\theta}\frac{M}{P}\right]^{\frac{1}{\gamma}}.
$$

Re-arranging and solving for $\frac{P_{ij}}{P}$:

$$
\begin{aligned}
L_{ij}^{\gamma} &= \frac{\alpha}{(1-\alpha)}\left(\frac{P_{ij}}{P}\right)^{-\theta}\frac{M}{P} \Leftrightarrow \\
\left(\frac{P_{ij}}{P}\right)^{-\theta} &= \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{-1}L_{ij}^{\gamma} \Leftrightarrow \\
\frac{P_{ij}}{P} &= \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{\frac{1}{\theta}}L_{ij}^{-\frac{\gamma}{\theta}}.
\end{aligned}
$$

Raising both sides to the power of $1-\theta$:

$$
\frac{P_{ij}^{1-\theta}}{P^{1-\theta}} = \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{\frac{1-\theta}{\theta}}L_{ij}^{-\frac{\gamma(1-\theta)}{\theta}}.
$$

Integrating both sides over $(0,1)$:

$$\frac{\int_0^1 P_{ij}^{1-\theta}dj}{P^{1-\theta}} = \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{\frac{1-\theta}{\theta}} \int_0^1 L_{ij}^{-\frac{\gamma(1-\theta)}{\theta}} dj \Leftrightarrow$$

$$1 = \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{\frac{1-\theta}{\theta}} \int_0^1 L_{ij}^{-\frac{\gamma(1-\theta)}{\theta}} dj \Leftrightarrow$$

$$1 = \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{-\left(\frac{\theta-1}{\theta}\right)} \int_0^1 L_{ij}^{\frac{\gamma(\theta-1)}{\theta}} dj \Leftrightarrow$$

$$\int_0^1 L_{ij}^{\frac{\gamma(\theta-1)}{\theta}} dj = \left[\frac{\alpha}{1-\alpha}\frac{M}{P}\right]^{\frac{\theta-1}{\theta}}$$

I next define aggregate labour demand, L^D:

$$L^D = \left(\int_0^1 L_{ij}^{\frac{\gamma(\theta-1)}{\theta}} dj\right)^{\frac{\theta}{\gamma(\theta-1)}}$$

I then obtain:

$$\left(L^D\right)^{\frac{\gamma(\theta-1)}{\theta}} = \left[\frac{\alpha}{1-\alpha}\frac{M}{P}\right]^{\frac{(\theta-1)}{\theta}}$$

and thus

$$L^D = \left[\frac{\alpha}{1-\alpha}\frac{M}{P}\right]^{\frac{1}{\gamma}}$$

Since the total labour force equals the mass of households, the labour force has mass one. The aggregate unemployment rate is therefore given by:

$$u = 1 - (L^D + L^G).$$

Substituting for aggregate demand from the private and public sectors, I obtain:

$$u = 1 - \left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{\frac{1}{\gamma}} - G.$$

To obtain an expression for the aggregate price level, I need to aggregate the price setting rules of each firm:

$$\left(\frac{P_{ij}}{P}\right) = \zeta \left(\frac{W_i}{P}\right)^{\gamma\phi} \left(\frac{M}{P}\right)^{(1-\gamma)\phi}.$$

Raising both sides to the power of $1-\theta$ implies:

$$\frac{P_{ij}^{1-\theta}}{P^{1-\theta}} = \zeta^{1-\theta} \left(\frac{W_i}{P}\right)^{\gamma\phi(1-\theta)} \left(\frac{M}{P}\right)^{(1-\gamma)\phi(1-\theta)}.$$

Averaging over the intervals covered by each union implies:

$$\frac{1}{P^{1-\theta}}\frac{1}{N}\sum_{i=0}^{N-1}\frac{\int_{\frac{i}{N}}^{\frac{i+1}{N}} P_{ij}^{1-\theta}dj}{\int_{\frac{i}{N}}^{\frac{i+1}{N}} dj} = \zeta^{1-\theta}\left(\frac{1}{P}\right)^{\gamma\phi(1-\theta)} \frac{1}{N}\sum_{i=0}^{N-1} W_i^{\gamma\phi(1-\theta)} \left(\frac{M}{P}\right)^{(1-\gamma)\phi(1-\theta)}.$$

Note that the left-hand side can be written:

$$
\begin{aligned}
\frac{1}{P^{1-\theta}}\frac{1}{N}\sum_{i=0}^{N-1}\frac{\int_{\frac{i}{N}}^{\frac{i+1}{N}}P_{ij}^{1-\theta}dj}{\int_{\frac{i}{N}}^{\frac{i+1}{N}}dj} &= \frac{1}{P^{1-\theta}}\frac{1}{N}\sum_{i=0}^{N-1}\frac{\int_{\frac{i}{N}}^{\frac{i+1}{N}}P_{ij}^{1-\theta}dj}{[j]_{\frac{i}{N}}^{\frac{i+1}{N}}} = \\
\frac{1}{P^{1-\theta}}\frac{1}{N}\sum_{i=0}^{N-1}\frac{\int_{\frac{i}{N}}^{\frac{i+1}{N}}P_{ij}^{1-\theta}dj}{1/N} &= \frac{1}{P^{1-\theta}}\sum_{i=0}^{N-1}\int_{\frac{i}{N}}^{\frac{i+1}{N}}P_{ij}^{1-\theta}dj = \frac{\int_0^1 P_{ij}^{1-\theta}dj}{P^{1-\theta}} = 1.
\end{aligned}
$$

I thus obtain:

$$
\begin{aligned}
1 &= \zeta^{1-\theta}\left(\frac{1}{P}\right)^{\gamma\phi(1-\theta)}\frac{1}{N}\sum_{i=0}^{N-1}W_i^{\gamma\phi(1-\theta)}\left(\frac{M}{P}\right)^{(1-\gamma)\phi(1-\theta)} \Leftrightarrow \\
1 &= \zeta^{1-\theta}\left(\frac{1}{P}\right)^{\gamma\phi(1-\theta)}W^{\gamma\phi(1-\theta)}\left(\frac{M}{P}\right)^{(1-\gamma)\phi(1-\theta)}.
\end{aligned}
$$

where W is the aggregate wage index defined as:

$$
W = \left(\frac{1}{N}\sum_{i=0}^{N-1}W_i^{\gamma\phi(1-\theta)}\right)^{\frac{1}{\gamma\phi(1-\theta)}}.
$$

By noting that $\gamma\phi(1-\theta) = 1-\theta\phi$, I can write:

$$
W = \left(\frac{1}{N}\sum_{i=0}^{N-1}W_i^{1-\theta\phi}\right)^{\frac{1}{1-\theta\phi}}.
$$

Simplifying:

$$
\begin{aligned}
1 = \zeta^{1-\theta}\left(\frac{1}{P}\right)^{\gamma\phi(1-\theta)}W^{\gamma\phi(1-\theta)}\left(\frac{M}{P}\right)^{(1-\gamma)\phi(1-\theta)} &\Leftrightarrow \\
P^{(\gamma\phi(1-\theta)+(1-\gamma)\phi(1-\theta))} = \zeta^{1-\theta}W^{\gamma\phi(1-\theta)}M^{(1-\gamma)\phi(1-\theta)} &\Leftrightarrow \\
P^{\phi(1-\theta)} = \zeta^{1-\theta}W^{\gamma\phi(1-\theta)}M^{(1-\gamma)\phi(1-\theta)}&.
\end{aligned}
$$

Simplifying this expression I obtain:

$$
P = \kappa W^{\gamma}M^{(1-\gamma)},
$$

where $\kappa \equiv \left[\frac{\alpha}{1-\alpha}\right]^{(1-\gamma)}\left[\frac{\theta}{\gamma(\theta-1)}\right]^{\gamma}$.

Monetary Policy

The objective function of the central bank is given by:

$$
M^{\varphi}P^{1-\varphi} = c.
$$

If and only if

$$
M = c^{\frac{1}{\varphi}}P^{\frac{\varphi-1}{\varphi}}.
$$

Substituting for the equilibrium price level:

$$
M = c^{\frac{1}{\varphi}}P^{\frac{\varphi-1}{\varphi}} = c^{\frac{1}{\varphi}}\left[\kappa W^{\gamma}M^{(1-\gamma)}\right]^{\frac{\varphi-1}{\varphi}} = c^{\frac{1}{\varphi}}\kappa^{\frac{\varphi-1}{\varphi}}W^{\gamma\frac{\varphi-1}{\varphi}}M^{(1-\gamma)\frac{\varphi-1}{\varphi}}.
$$

This implies:

$$M^{1-(1-\gamma)\frac{\varphi-1}{\varphi}} = c^{\frac{1}{\varphi}}\kappa^{\frac{\varphi-1}{\varphi}}W^{\frac{\gamma(\varphi-1)}{\varphi}} \Leftrightarrow$$
$$M^{\frac{\varphi-(1-\gamma)(\varphi-1)}{\varphi}} = c^{\frac{1}{\varphi}}\kappa^{\frac{\varphi-1}{\varphi}}W^{\frac{\gamma(\varphi-1)}{\varphi}} \Leftrightarrow$$
$$M = \left[c^{\frac{1}{\varphi}}\kappa^{\frac{\varphi-1}{\varphi}}W^{\frac{\gamma(\varphi-1)}{\varphi}}\right]^{\frac{\varphi}{\varphi+(1-\gamma)(1-\varphi)}} \Leftrightarrow$$
$$M = c^{\frac{1}{\varphi+(1-\gamma)(1-\varphi)}}\kappa^{\frac{\varphi-1}{\varphi+(1-\gamma)(1-\varphi)}}W^{-\frac{\gamma(1-\varphi)}{\varphi+(1-\gamma)(1-\varphi)}}.$$

Thus:

$$M = \widetilde{\kappa}W^{-\gamma(1-\varphi)\varpi}$$

where $\widetilde{\kappa} \equiv c^{\varpi}\kappa^{-(1-\varphi)\varpi}$ and $\varpi = \frac{1}{\varphi+(1-\gamma)(1-\varphi)} > 0$. Substituting this expression into the objective function of the central bank gives the following expression for the equilibrium price level:

$$\begin{aligned} P &= c^{\frac{1}{1-\varphi}}M^{\frac{-\varphi}{1-\varphi}} = c^{\frac{1}{1-\varphi}}\left[\widetilde{\kappa}W^{-\gamma(1-\varphi)\varpi}\right]^{\frac{-\varphi}{1-\varphi}} \\ &= c^{\frac{1}{1-\varphi}}\widetilde{\kappa}^{\frac{-\varphi}{1-\varphi}}W^{\frac{\varphi}{1-\varphi}\gamma(1-\varphi)\varpi} = c^{\frac{1}{1-\varphi}}\widetilde{\kappa}^{\frac{-\varphi}{1-\varphi}}W^{\varphi\gamma\varpi}. \end{aligned}$$

Thus, the aggregate price level can be written:

$$P = \frac{W^{\gamma\varphi\varpi}}{\widetilde{\chi}},$$

where $\widetilde{\chi} \equiv c^{-(1-\gamma)\varpi}\left(\frac{\alpha}{1-\alpha}\right)^{-(1-\gamma)\varphi\varpi}\left(\frac{\theta}{\gamma(\theta-1)}\right)^{-\gamma\varphi\varpi}$ and $\varpi = \frac{1}{\varphi+(1-\gamma)(1-\varphi)}$.

The aggregate unemployment rate is given by:

$$\begin{aligned} u &= 1-\left[\frac{\alpha}{(1-\alpha)}\frac{M}{P}\right]^{\frac{1}{\gamma}} - L^G = 1-\left[\frac{\alpha}{(1-\alpha)}\kappa^{-\varpi}\left(\frac{W}{c}\right)^{-\gamma\varpi}\right]^{\frac{1}{\gamma}} - G \\ &= 1-\delta\left(\frac{W}{c}\right)^{-\varpi} - G. \end{aligned}$$

where $\delta \equiv \left(\frac{\alpha}{(1-\alpha)}\kappa^{-\varpi}\right)^{\frac{1}{\gamma}} = \left(\frac{\alpha}{1-\alpha}\right)^{\varphi\varpi}\left(\frac{\theta}{\gamma(\theta-1)}\right)^{-\varpi}$.

The Unemployment Rate facing Union i

The expression for unemployment facing union i is:

$$u_i = 1-\vartheta\left(\frac{M}{P}\right)^{\phi}\left(\frac{W_i}{P}\right)^{-\phi\theta} - G.$$

Moreover:

$$\frac{W_i}{P} = \kappa^{-\varphi\varpi}\frac{W_i}{c}\left(\frac{W}{c}\right)^{-\gamma\varphi\varpi},$$

and

$$\frac{M}{P} = \kappa^{-\varpi}\left(\frac{W}{c}\right)^{-\gamma\varpi}.$$

Substituting this into the expression for the unemployment rate implies:

$$\begin{aligned} u_i &= 1-\vartheta\left(\frac{M}{P}\right)^{\phi}\left(\frac{W_i}{P}\right)^{-\phi\theta}-G \\ &= 1-\vartheta\left(\kappa^{-\varpi}\left(\frac{W}{c}\right)^{-\gamma\varpi}\right)^{\phi}\left(\kappa^{-\varphi\varpi}\frac{W_i}{c}\left(\frac{W}{c}\right)^{-\gamma\varphi\varpi}\right)^{-\phi\theta}-G \\ &= 1-\delta\left(\frac{W_i}{c}\right)^{-\phi\theta}\left(\frac{W}{c}\right)^{-\gamma\varpi\phi(1-\theta\varphi)}-G, \end{aligned}$$

where $\delta \equiv \left(\frac{\alpha}{1-\alpha}\right)^{\varphi\varpi}\left(\frac{\theta}{\gamma(\theta-1)}\right)^{-\varpi}$.

Real Wages

The consumer real wage is given by:

$$\frac{W}{P} = \widetilde{\chi}W\left(W^{\gamma\varphi\varpi}\right)^{-1} = \kappa^{-\varphi\varpi}\left(\frac{W}{c}\right)^{(1-\gamma)\varpi}.$$

And the real wage obtained by the members of union i can be written:

$$\frac{W_i}{P} = \kappa^{-\varphi\varpi}\frac{W_i}{c}\left(\frac{W}{c}\right)^{-\gamma\varphi\varpi}.$$