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WHAT ÖSTERBERG'S POPULATION THEORY HAS IN COMMON WITH PLATO'S

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1. Introduction

Jan Österberg is one of the pioneers in the field of population ethics. He started thinking about this issue already in the late 60s and he has developed one of the most original and interesting population axiologies.[1] I've discussed the problems and drawbacks of Österberg's theory elsewhere, and I don't think that this is the place and time to discuss them again.[2] Rather, I shall show that Österberg's theory has a feature in common with the population axiologies of such luminaries like Plato, Aristotle, Kant, Nietzsche, Wittgenstein and Heidegger, had they developed such a theory: None of these theories simultaneously satisfy five weak adequacy conditions. We shall show this by proving that no population axiology satisfies these five conditions. As a fringe benefit, this theorem also shows that the on-going project of constructing an acceptable population axiology has very gloomy prospects. [3]

2. The basic structure

For the purpose of proving the theorem, it will be useful to state some definitions and assumptions, and introduce some notational conventions. A *life* is individuated by the person whose life it is and the kind of life it is. A *population*

[1] See Österberg (1992) and (1996).

[2] See Arrhenius (2000a), ch. 8, section 8.5.

[3] This theorem first appeared in Arrhenius (2000a). It is a development of the results presented in Arrhenius (1999ab, 2000b). The difference between the present theorem and my earlier contributions, and those of Parfit (1984), Ng (1989), and Blackorby and Donaldson (1991), is that it invokes intuitively more compelling and logically weaker conditions (for example, it doesn't involve any version of the controversial Mere Addition Principle), weaker measurement and ordering assumptions (a quasi-ordering of lives and populations is sufficient), and less undefined welfare concepts (for example, we don't invoke concepts such as "very high welfare" or "a life barely worth living").

is a finite set of lives in a possible world.[4] We shall assume that for any natural number *n* and any welfare level $\mathbf{X}$, there is a possible population of *n* people with welfare $\mathbf{X}$. Two populations are identical if and only if they consist of the same lives. Since the same person can exist (be instantiated) and lead the same kind of life in many different possible worlds, the same life can exist in many possible worlds. Moreover, since two populations are identical exactly if they consist of the same lives, the same population can exist in many possible worlds. A *population axiology* is an "at least as good as" quasi-ordering of all possible populations, that is, a reflexive, transitive, but not necessarily complete ordering of populations in regard to their goodness.

A, B, C,… A_1, A_2,…, A_n, $A \cup B$, and so on, denote populations of finite size. The number of lives in a population X (X's population size) is given by the function *N*(X). We shall adopt the convention that populations represented by different letters, or the same letter but different indexes, are pairwise disjoint.

The relation "*has at least as high welfare as*" quasi-orders (reflexive, transitive, but not necessarily complete) the set $\mathbf{L}$ of all possible lives. A life p_1 has higher welfare than another life p_2 if and only if p_1 has at least as high welfare as p_2 and it is not the case that p_2 has at least as high welfare as p_1. p_1 has the same welfare as p_2 if and only if p_1 has at least as high welfare as p_2 and p_2 has at least as high welfare as p_1. We shall say that a life has *neutral welfare* if and only if it has the same welfare as a life without any good or bad welfare-components, and that a life has *positive* (*negative*) welfare if and only if it has higher (lower) welfare than a life with neutral welfare.

By a *welfare level* $\mathbf{A}$ we shall mean a set such that if a life *a* is in $\mathbf{A}$, then a life *b* is in $\mathbf{A}$ if and only if *b* has the same welfare as *a*. In other words, a welfare level is an equivalence class on $\mathbf{L}$. Let *a** be a life which is representative of the welfare level $\mathbf{A}$. We shall say that a welfare level $\mathbf{A}$ is higher (lower, the same) than (as) a level $\mathbf{B}$ if and only if *a** has higher (lower, the same) welfare than (as) *b**; that a welfare level $\mathbf{A}$ is positive (negative, neutral) if and only if *a** has positive (negative, neutral) welfare; and that a life *b* has welfare below (above, at) $\mathbf{A}$ if and only if *b* has higher (lower, the same) welfare than (as) *a**.

We shall assume that *Discreteness* is true of the set of all possible lives $\mathbf{L}$ or some subset of $\mathbf{L}$:

[4] For some possible constraints on possible populations, see Arrhenius (2000a), ch. 2.

Discreteness: For any pair of welfare levels X and Y, X higher than Y, the set consisting of all welfare levels Z such that X is higher than Z, and Z is higher than Y, has a finite number of members.

The statement of the informal version of some of the adequacy conditions below, for example the Quantity Condition, involve the not so exact relation "slightly higher welfare than". In the exact statements of those adequacy conditions, we shall instead make use of two consecutive welfare levels, that is, two welfare levels such that there is no welfare level in between them. Discreteness ensures that there are such welfare levels. Intuitively speaking, if A and B are two consecutive welfare levels, A higher than B, then A is just slightly higher than B. More importantly, the intuitive plausibility of the adequacy conditions is preserved. Of course, this presupposes that the order of welfare levels is fine-grained, which is exactly what is suggested by expressions such as "Jan is slightly better off than Gert" and the like.[5] Notice that Discreteness doesn't exclude the view that for any welfare level, there is a higher and a lower welfare level (compare with the natural numbers).

Given Discreteness, we can index welfare levels with integers in a natural manner. Discreteness in conjunction with the existence of a neutral welfare level and a quasi-ordering of lives implies that there is at least one positive welfare level in L such that there is no lower positive welfare level.[6] Let W_1, W_2, W_3,… and so forth represent positive welfare levels, starting with one of the positive welfare level for which there is no lower positive one, such that for any pair of welfare levels W_n and W_{n+1}, W_{n+1} is higher than W_n, and there is no welfare level X such that W_{n+1} is higher than X, and X is higher than W_n. Analogously, let W_{-1}, W_{-2}, W_{-3},… and so on represent negative welfare levels. The neutral welfare level is represented by W_0.

A *welfare range* R(x, y) is a union of at least *three* welfare levels defined by two welfare levels W_x and W_y, $x < y$, such that for any welfare level W_z, W_z is a subset of R(x, y) if and only if $x \leq z \leq y$.[7] We shall say that a welfare range R(x, y) is higher (lower) than another range R(z, w) if and only if $x > w$ ($y < z$); that a welfare range R(x, y) is positive (negative) if and only if $x > 0$ ($y < 0$); and

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[5] For a defense of the plausibility of Discreteness, see Arrhenius (2000a), ch 10, section 9.

[6] There might be more than one since we only have an quasi-ordering of lives, that is, there might be lives and thus welfare levels which are incommensurable in regard to welfare.

[7] The reason for restricting welfare ranges to unions of at least three welfare levels, as opposed to at least two welfare levels, is that this restriction allows us to simplify the exact statements of the adequacy conditions.

that a life *p* has welfare above (below, in) $\mathcal{R}(x, y)$ if and only if *p* is in some $\mathcal{W}_z$ such that $z > y$ ($z < x$, $y \geq z \geq x$).

3. Adequacy conditions

We shall make use of the following five adequacy conditions:

> *The Egalitarian Dominance Condition*: If population A is a perfectly equal population of the same size as population B, and every person in A has higher welfare than every person in B, then A is better than B, other things being equal.
>
> *The Egalitarian Dominance Condition (exact formulation)*: For any populations A and B, $N(A)=N(B)$, and any welfare level $\mathcal{W}_x$, **if** all members of B have welfare below $\mathcal{W}_x$, and $A\subset\mathcal{W}_x$, **then** A is better than B, other things being equal.
>
> *The General Non-Extreme Priority Condition*: There is a number *n* of lives such that for any population X, and any welfare level $\mathcal{A}$, a population consisting of the X-lives, *n* lives with very high welfare, and one life with welfare $\mathcal{A}$, is at least as good as a population consisting of the X-lives, *n* lives with very low positive welfare, and one life with welfare slightly above $\mathcal{A}$, other things being equal.
>
> *The General Non-Extreme Priority Condition (exact formulation)*: For any $\mathcal{W}_z$, there is a positive welfare level $\mathcal{W}_u$, and a positive welfare range $\mathcal{R}(1, y)$, $u > y$, and a number of lives $n > 0$ such that **if** $A\subset\mathcal{W}_x$, $x \geq u$, $B\subset\mathcal{R}(1, y)$, $N(A)=N(B)=n$, $C\subset\mathcal{W}_z$, $D\subset\mathcal{W}_{z+1}$, $N(C)= N(D)=1$, **then,** for any E, $A\cup C\cup E$ is at least as good as $B\cup D\cup E$, other things being equal.
>
> *The Non-Elitism Condition*: For any triplet of welfare levels $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, $\mathcal{A}$ slightly higher than $\mathcal{B}$, and $\mathcal{B}$ higher than $\mathcal{C}$, and for any one-life population A with welfare $\mathcal{A}$, there is a population C with welfare $\mathcal{C}$, and a population B of the same size as $A\cup C$ and with welfare $\mathcal{B}$, such that for any population X consisting of lives with welfare ranging from $\mathcal{C}$ to $\mathcal{A}$, $B\cup X$ is at least as good as $A\cup C\cup X$, other things being equal.
>
> *The Non-Elitism Condition (exact formulation)*: For any welfare levels $\mathcal{W}_x$, $\mathcal{W}_y$, $x\text{-}1 > y$, there is a number of lives $n > 0$ such that **if** $A\subset\mathcal{W}_x$, $N(A)=1$, $B\subset\mathcal{W}_y$, $N(B)=n$, and $C\subset\mathcal{W}_{x-1}$, $N(C)=n+1$, **then**, for any

$D \subset \mathcal{R}(y, x)$, $C \cup D$ is at least as good as $A \cup B \cup D$, other things being equal.

The Weak Non-Sadism Condition: There is a negative welfare level and a number of lives at this level such that an addition of any number of people with positive welfare is at least as good as an addition of the lives with negative welfare, other things being equal.

The Weak Non-Sadism Condition (exact formulation): There is a welfare level $\mathcal{W}_x$, $x < 0$, and a number of lives n, such that **if** $A \subset \mathcal{W}_x$, $N(A)=n$, $B \subset \mathcal{W}_y$, $y > 0$, **then**, for any population C, $B \cup C$ is at least as good as $A \cup C$, other things being equal.

The Weak Quality Addition Condition: For any population X, there is at least one perfectly equal population with very high welfare such that its addition to X is at least as good as an addition of any population with very low positive welfare to X, other things being equal.

The Weak Quality Addition Condition (exact formulation): For any population C, there are two positive welfare ranges $\mathcal{R}(x, w)$ and $\mathcal{R}(1, y)$, $x > y$, and a population size n such that **if** $A \subset \mathcal{W}_z$, $z \geq x$, $N(A)=n$, $B \subset \mathcal{R}(1, y)$, **then** $A \cup C$ is at least as good as $B \cup C$, other things being equal.

4. The impossibility theorem

The Impossibility Theorem: There is no population axiology which satisfies the Egalitarian Dominance, the Non-Elitism, the General Non-Extreme Priority, the Weak Non-Sadism, and the Weak Quality Addition Condition.

Proof. We shall show that the contrary assumption leads to a contradiction. We shall first prove two lemmas to the effect that the Non-Elitism and the General Non-Extreme Priority Condition each imply another condition. Then we shall show that there is no population axiology which satisfies these two new conditions in conjunction with the Egalitarian Dominance, the Weak Non-Sadism, and the Weak Quality Addition Condition.

4.1. Lemma 1

Lemma 1: The Non-Elitism Condition implies Condition β:

Condition β: For any triplet $\mathbb{W}_x$, $\mathbb{W}_y$, $\mathbb{W}_z$ of welfare levels, $x > y > z$, and any number of lives $n > 0$, there is a number of lives $m > n$ such that **if** $A \subset \mathbb{W}_x$, $N(A)=n$, $B \subset \mathbb{W}_z$, $N(B)=m$, and $C \subset \mathbb{W}_y$, $N(C)=m+n$, **then,** for any $D \subset \mathbb{R}(z, y+1)$, $C \cup D$ is at least as good as $A \cup B \cup D$, other things being equal.

We shall prove lemma 1 by first proving

Lemma 1.1: The Non-Elitism Condition entails Condition α.

Condition α: For any welfare levels $\mathbb{W}_x$, $\mathbb{W}_y$, $x\text{-}1 > y$, and for any number of lives $n > 0$, there is a number of lives $m \geq n$ such that **if** $A \subset \mathbb{W}_x$, $N(A)=n$, $B \subset \mathbb{W}_y$, $N(B)=m$, $C \subset \mathbb{W}_{x-1}$, $N(C)=m+n$, **then**, for any $D \subset \mathbb{R}(y, x)$, $C \cup D$ is at least as good as $A \cup B \cup D$, other things being equal.

Proof: Let

(1) $\mathbb{W}_x$ and $\mathbb{W}_y$ be any welfare levels such that $x\text{-}1 > y$;

(2) n be any number of lives such that $n > 0$;

(3) $p > 0$ be a number which satisfies the Non-Elitism Condition for $\mathbb{W}_x$ and $\mathbb{W}_y$.

Let $A_1, \ldots, A_{n+1}$, $B_1, \ldots, B_{n+1}$, and $C_0, \ldots, C_n$, be any three sequences of populations satisfying

(4) $A_i \subset \mathbb{W}_x$; $N(A_i)=1$ for all i, $1 \leq i \leq n$; $A_{n+1}=\varnothing$;

(5) $B_i \subset \mathbb{W}_y$; $N(B_i)=p$, for all i, $1 \leq i \leq n$; $B_{n+1}=\varnothing$;

(6) $C_i \subset \mathbb{W}_{x-1}$; $N(C_i)=p+1$, for all i, $1 \leq i \leq n$; $C_0=\varnothing$.

Finally, let

(7) D be any population such that $D \subset \mathbb{R}(y, x)$.

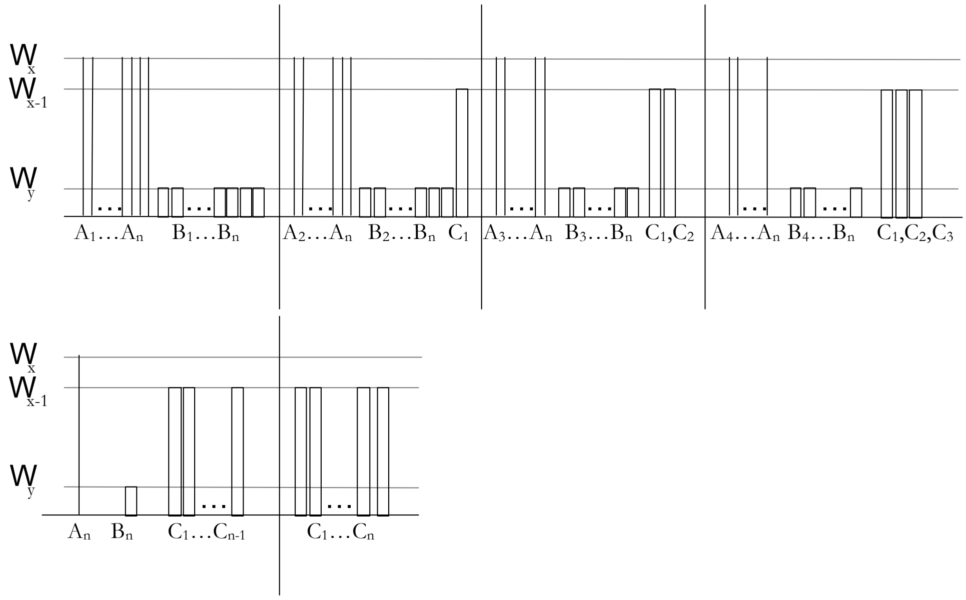

Diagram 1

The above diagram shows a selection of the involved populations in a case where $n \geq 6$. Dots in between two blocks indicate that there is a number of same sized blocks which have been omitted from the diagram. Population D is omitted throughout.

Since $\mathbb{W}_x$ and $\mathbb{W}_y$ can be any pair of welfare levels separated by at least one welfare level, and D can be any population consisting of lives with welfare ranging from $\mathbb{W}_y$ to $\mathbb{W}_x$, and $N(A_1 \cup \ldots \cup A_n) = n$ (by (4)) can be any number of lives greater than zero, and $N(B_1 \cup \ldots \cup B_n) = np \geq n$ (by (5)), we can show that lemma 1.1 is true by showing that $C_1 \cup \ldots \cup C_n \cup D$ is at least as good as $A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_n \cup D$. This suffices since $A_1, \ldots, A_{n+1}$, $B_1, \ldots, B_{n+1}$, $C_1, \ldots, C_n$, and D are arbitrary populations satisfying (4)-(7).

It follows from (3)-(6) and the Non-Elitism Condition that

(8) $C_i \cup E$ is at least as good as $A_i \cup B_i \cup E$ for all i, $1 \leq i \leq n$ and any $E \subset \mathbb{R}(y, x)$

and from (4)-(7) that

(9) $A_{i+1} \cup \ldots \cup A_{n+1} \cup B_{i+1} \cup \ldots \cup B_{n+1} \cup C_0 \cup \ldots \cup C_{i-1} \cup D \subset \mathbb{R}(y, x)$ for all i, $1 \leq i \leq n$.

Letting $E = A_{i+1} \cup \ldots \cup A_{n+1} \cup B_{i+1} \cup \ldots \cup B_{n+1} \cup C_0 \cup \ldots \cup C_{i-1} \cup D$, (8) and (9) imply that

(10) $C_i \cup [A_{i+1} \cup \ldots \cup A_{n+1} \cup B_{i+1} \cup \ldots \cup B_{n+1} \cup C_0 \cup \ldots \cup C_{i-1} \cup D]$ is at least as good as $A_i \cup B_i \cup [A_{i+1} \cup \ldots \cup A_{n+1} \cup B_{i+1} \cup \ldots \cup B_{n+1} \cup C_0 \cup \ldots \cup C_{i-1} \cup D]$ for all i, $1 \leq i \leq n$ (see Diagram 1).

Transitivity and (10) yield that

(11) $C_n \cup A_{n+1} \cup B_{n+1} \cup C_0 \cup \ldots \cup C_{n-1} \cup D$ is at least as good as $A_1 \cup B_1 \cup A_2 \cup \ldots \cup A_{n+1} \cup B_2 \cup \ldots \cup B_{n+1} \cup C_0 \cup D$

and since $A_{n+1}=B_{n+1}=C_0=\varnothing$ (4-6), line (11) is equivalent to (see *Diagram 1*)

(12) $C_1 \cup \ldots \cup C_n \cup D$ is at least as good as $A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_n \cup D$.

Q.E.D.

To show that Lemma 1 is true, we now need to prove

Lemma 1.2: Condition α entails Condition β.

Proof. Let

(1) $\mathbb{W}_x$, $\mathbb{W}_y$, $\mathbb{W}_z$ be any three welfare levels such that $x > y > z$;

(2) $r = x\text{-}y$.

Let $A_1, \ldots, A_{r+1}$ and $B_1, \ldots, B_{r+1}$ be any two sequences of populations, $m_0, \ldots, m_r$ any sequence of integers, and f a function satisfying

(3) $m_0 > 0$;

(4) $f(m_i)=m_0+m_1+\ldots+m_i$, for all i, $0 \leq i \leq r$,

(5) $m_i \geq f(m_{i-1})$ satisfies Condition α for $\mathbb{W}_{x-(i-1)}$, $\mathbb{W}_z$, and $f(m_{i-1})$ for all i, $1 \leq i \leq r$;

(6) $A_i \subset \mathbb{W}_{x-(i-1)}$, $N(A_i)=f(m_{i-1})$ for all i, $1 \leq i \leq r+1$;

(7) $B_i \subset \mathbb{W}_z$, $N(B_i)=m_i$, for all i, $1 \leq i \leq r$; $B_{r+1}=\varnothing$.

Finally, let

(8) D be any population such that $D \subset \mathbb{R}(z, y+1)$;

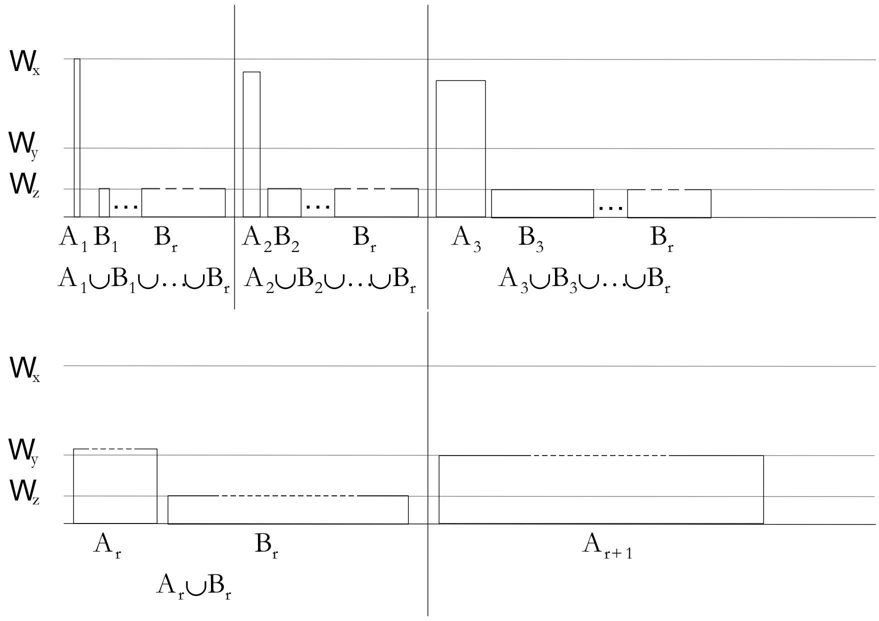

Diagram 2

The above diagram shows a selection of the involved populations in a case where $r \geq 4$. Population D is omitted throughout.

We can conclude from (3)-(7) that $N(B_1 \cup \ldots \cup B_r) > m_0 = N(A_1)$. Consequently, since $\mathbb{W}_x$, $\mathbb{W}_y$, and $\mathbb{W}_z$ can be any welfare levels such that $x > y > z$, and D can be any population consisting of lives with welfare ranging from $\mathbb{W}_z$ to $\mathbb{W}_{y+1}$, we can show that Condition α implies Condition β by showing that $A_{r+1} \cup D$ is at least as good as $A_1 \cup B_1 \cup \ldots \cup B_r \cup D$. This suffices since $A_1, \ldots, A_{r+1}$, $B_1, \ldots, B_r$, and D are arbitrary populations satisfying (6)-(8).

From (3)-(7) and Condition α, it follows that

(9) $A_{i+1} \cup E$ is at least as good as $A_i \cup B_i \cup E$ for all i, $1 \leq i \leq r$ and any $E \subset \mathcal{R}(z, y+1)$.

and from (7) and (8) that

(10) $B_{i+1} \cup \ldots \cup B_{r+1} \cup D \subset \mathcal{R}(z, y+1)$ for all i, $1 \leq i \leq r$.

Consequently, letting $E = B_{i+1} \cup \ldots \cup B_{r+1} \cup D$, (9) and (10) imply that

(11) $A_{i+1} \cup [B_{i+1} \cup \ldots \cup B_{r+1} \cup D]$ is at least as good as $A_i \cup B_i \cup [B_{i+1} \cup \ldots \cup B_{r+1} \cup D]$ for all i, $1 \leq i \leq r$ (see *Diagram 2*).

Transitivity and (11) yield that

(12) $A_{r+1} \cup B_{r+1} \cup D$ is at least as good as $A_1 \cup B_1 \cup \ldots \cup B_{r+1} \cup D$

and since $B_{r+1} = \varnothing$ (7), line (12) is equivalent to (see *Diagram 2*)

(13) $A_{r+1} \cup D$ is at least as good as $A_1 \cup B_1 \cup \ldots \cup B_r \cup D$. Q.E.D.

It follows trivially from lemma 1.1 and 1.2 that lemma 1 is true. Q.E.D.

4.2. Lemma 2

Lemma 2: The General Non-Extreme Priority Condition implies Condition δ.

Condition δ: For any W_z, $z < 0$, and any number of lives $m > 0$, there is a positive welfare level W_u, and a positive welfare range $R(1, y)$, $u > y$, and a number of lives $n > 0$ such that **if** $A \subset W_x$, $x \geq u$, $B \subset R(1, y)$, $N(A)=N(B)=n$, $C \subset W_z$, $D \subset W_3$, $N(C)=N(D)=m$, **then,** for any E, $A \cup C \cup E$ is at least as good as $B \cup D \cup E$, other things being equal.

We shall prove lemma 2 by first proving

Lemma 2.1: The General Non-Extreme Priority Condition implies Condition χ.

Condition χ: For any W_z, $z < 0$, there is a positive welfare level W_u, and a positive welfare range $R(1, y)$, $u > y$, and a number of lives $n > 0$ such that **if** $A \subset W_x$, $x \geq u$, $B \subset R(1, y)$, $N(A)=N(B)=n$, $C \subset W_z$, $D \subset W_3$, $N(C)=N(D)=1$, **then,** for any E, $A \cup C \cup E$ is at least as good as $B \cup D \cup E$, other things being equal.

Proof: Let

(1) W_z be any welfare level such that $z < 0$;

(2) $r=3-z$;

(3) W_{u_i} be a positive welfare level, $R(1, v_i)$ be a positive welfare range, and n_i a number of lives which satisfy the General Non-Extreme Priority Condition for $W_{z+(i-1)}$ for all i, $1 \leq i \leq r$;

(4) W_u be a welfare level such that u equals the maximal element in $\{u_i: 1 \leq i \leq r\}$;

(5) W_x be a welfare level such that $x \geq u$;

(6) y be a number such that y equals the minimal element in $\{v_i: 1 \leq i \leq r\}$.

Let $A_1, \ldots, A_{r+1}$, $B_0, \ldots, B_r$, and $C_1, \ldots, C_{r+1}$, be any three sequences of populations satisfying

(7) $A_i \subset W_x$, $N(A_i)=n_i$, for all i, $1 \leq i \leq r$; $A_{r+1}=\varnothing$;

(8) $B_i \subset R(1, y)$, $N(B_i)=n_i$, for all i, $1 \leq i \leq r$; $B_0=\varnothing$;

(9) $C_i \subset W_{z+(i-1)}$, $N(C_i)=1$, for all i, $1 \leq i \leq r+1$.

Finally, let

(10) E be any population.

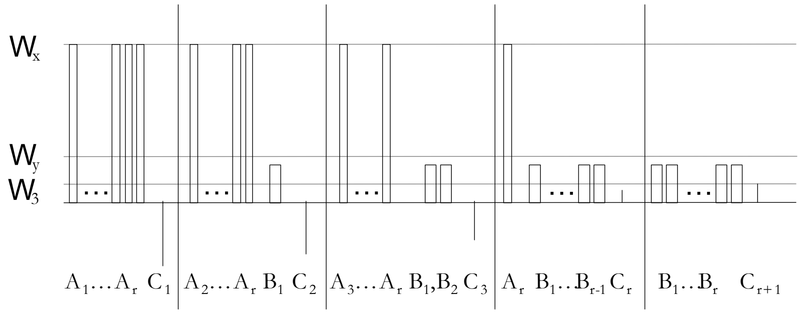

Diagram 3

The above diagram shows a selection of the involved populations in a case where $r \geq 4$. Population E is omitted throughout.

Since W_z can be any negative welfare level (by (1)), and W_x can be any welfare level at least as high as W_u (by (5)), and since it follows from (3) and (6) that $R(1, y)$ is a welfare range such that $u > y$, we can show that lemma 2.1 is true by showing that $A_1 \cup \ldots \cup A_r \cup C_1 \cup E$ is at least as good as $B_1 \cup \ldots \cup B_r \cup C_{r+1} \cup E$. This suffices since $A_1, \ldots, A_r$, $B_1, \ldots, B_r$, $C_1, \ldots, C_{r+1}$, and E are arbitrary populations satisfying (7)-(10).

The General Non-Extreme Priority Condition and (3)-(9) imply that

(11) $A_i \cup C_i \cup F$ is at least as good as $B_i \cup C_{i+1} \cup F$ for all i, $1 \leq i \leq r$ and any population F.

Letting $F = A_{i+1} \cup \ldots \cup A_{r+1} \cup B_o \cup \ldots \cup B_{i-1} \cup E$, it follows from (11) that

(12) $A_i \cup C_i \cup [A_{i+1} \cup \ldots \cup A_{r+1} \cup B_o \cup \ldots \cup B_{i-1} \cup E]$ is at least as good as $B_i \cup C_{i+1} \cup [A_{i+1} \cup \ldots \cup A_{r+1} \cup B_o \cup \ldots \cup B_{i-1} \cup E]$ for all i, $1 \leq i \leq r$ (see *Diagram 3*).

Transitivity and (12) yield that

(13) $A_1 \cup C_1 \cup A_2 \cup \ldots \cup A_{r+1} \cup B_o \cup E$ is at least as good as $B_r \cup C_{r+1} \cup A_{r+1} \cup B_o \cup \ldots \cup B_{r-1} \cup E$.

and since $A_{r+1} = B_0 = \varnothing$ (7-8), line (13) is equivalent to (see *Diagram 3*)

(14) $A_1 \cup \ldots \cup A_r \cup C_1 \cup E$ is at least as good as $B_1 \cup \ldots \cup B_r \cup C_{r+1} \cup E$.

Q.E.D.

To show that Lemma 2 is true, we now need to prove

Lemma 2.2: Condition χ implies Condition δ.
Proof: Let

(1) W_z be any welfare level such that $z < 0$;
(2) m be any number such that $m > 0$;
(3) W_u be a positive welfare level, $R(1, y)$ be a positive welfare range, and n a number of lives which satisfy Condition χ for W_z;
(4) W_x be a welfare level such that $x \geq u$.

Let $A_1,\ldots, A_{m+1}$, $B_0,\ldots, B_m$, $C_1,\ldots, C_{m+1}$, and $D_0,\ldots, C_m$, be any four sequences of populations satisfying

(5) $A_i \subset W_x$, $N(A_i)=n$, for all i, $1 \leq i \leq m$; $A_{m+1}=\varnothing$;
(6) $B_i \subset R(1, y)$, $N(B_i)=n$, for all i, $1 \leq i \leq m$; $B_0=\varnothing$;
(7) $C_i \subset W_z$, $N(C_i)=1$, for all i, $1 \leq i \leq m$; $C_{m+1}=\varnothing$;
(8) $D_i \subset W_3$, $N(D_i)=1$, for all i, $1 \leq i \leq m$; $D_0=\varnothing$.

Finally, let

(9) E be any population.

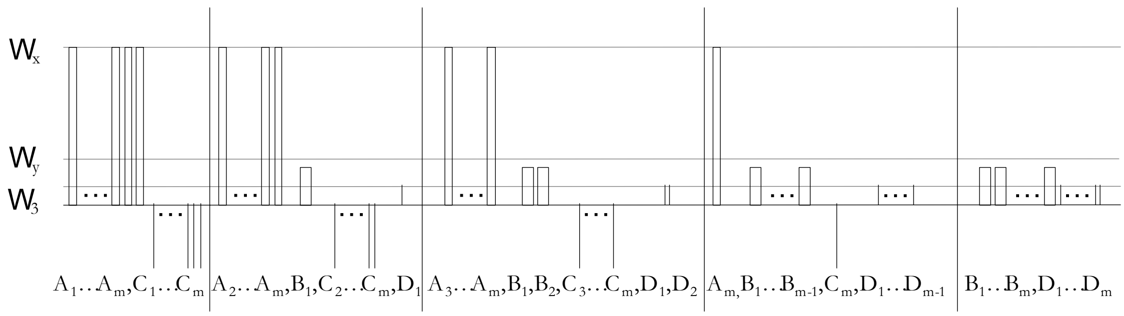

Diagram 4

The above diagram shows a selection of the involved populations in a case where $m \geq 4$. As before, population E is omitted throughout.

Since W_z can be any negative welfare level (by (1)), and W_x can be any welfare level at least as high as W_u (by (5)), and m can be any number of lives greater than zero, and $R(1, y)$ is a welfare range such that $u > y$, and n is a number greater than zero (by (3)), we can show that lemma 2.2 is true by showing that $A_1 \cup \ldots \cup A_m \cup C_1 \cup \ldots \cup C_m \cup E$ is at least as good as $B_1 \cup \ldots \cup B_m \cup D_1 \cup \ldots \cup D_m \cup E$. This suffices since $A_1,\ldots, A_m$, $B_1,\ldots, B_m$, $C_1,\ldots, C_m$, $D_1,\ldots, D_m$, and E are arbitrary populations satisfying (5)-(9).

It follows from (3)-(8) and Condition χ that

(10) $A_i \cup C_i \cup F$ is at least as good as $B_i \cup D_i \cup F$ for all i, $1 \le i \le m$, and any population F

which, for $F = A_{i+1} \cup \ldots \cup A_{m+1} \cup C_{i+1} \ldots \cup C_{m+1} \cup B_0 \cup \ldots \cup B_{i-1} \cup D_0 \ldots \cup D_{i-1} \cup E$, in turn implies

(11) $A_i \cup C_i \cup [A_{i+1} \cup \ldots \cup A_{m+1} \cup C_{i+1} \ldots \cup C_{m+1} \cup B_0 \cup \ldots \cup B_{i-1} \cup D_0 \ldots \cup D_{i-1} \cup E]$ is at least as good as $B_i \cup D_i \cup [A_{i+1} \cup \ldots \cup A_{m+1} \cup C_{i+1} \ldots \cup C_{m+1} \cup B_0 \cup \ldots \cup B_{i-1} \cup D_0 \ldots \cup D_{i-1} \cup E]$ for all i, $1 \le i \le m$ (see *Diagram 4*).

Transitivity and (11) yield that

(12) $A_1 \cup C_1 \cup A_2 \cup \ldots \cup A_{m+1} \cup C_2 \ldots \cup C_{m+1} \cup B_0 \cup D_0 \cup E$ is at least as good as $B_m \cup D_m \cup A_{m+1} \cup C_{m+1} \cup B_0 \cup \ldots \cup B_{m-1} \cup D_0 \ldots \cup D_{m-1} \cup E$

and since $A_{m+1} = B_0 = C_{m+1} = D_0 = \varnothing$ (by (5)-(8)), line (12) is equivalent to (see *Diagram 4*)

(13) $A_1 \cup \ldots \cup A_m \cup C_1 \ldots \cup C_m \cup E$ is at least as good as $B_1 \cup \ldots \cup B_m \cup D_1 \ldots \cup D_m \cup E$. Q.E.D.

It follows trivially from lemma 2.1 and 2.2 that lemma 2 is true. Q.E.D.

4.3. Lemma 3

Finally, we shall show that the impossibility theorem is true by proving

Lemma 3: There is no population axiology which satisfies the Egalitarian Dominance Condition, the Weak Non-Sadism Condition, the Weak Quality Addition Condition, Condition β, and Condition δ.

Proof. We show that the contrary assumption leads to a contradiction. Let

(1) $\mathcal{W}_z$ be a welfare level and m a population size which satisfy the Weak Non-Sadism Condition;

(2) $\mathcal{W}_u$ be a welfare level, $\mathcal{R}(1, y)$ a welfare range, and n a number of lives, which satisfy Condition δ for $\mathcal{W}_z$ and m;

(3) $B_1 \subset \mathcal{W}_3$, $B_2 \subset \mathcal{W}_3$, $N(B_1)=n$, $N(B_2)=m$;

(4) $\mathcal{R}(w, t)$ and $\mathcal{R}(1, v)$, $w > v$, be two welfare ranges, and p a population size, which satisfy the Weak Quality Addition Condition for B;

(5) Let $\mathcal{W}_x$ be a welfare level such that $x \ge w$ and $x \ge u$;

(6) $A \subset W_x$, $N(A)=p$;
(7) $H \subset W_x$, $N(H)=n$;
(8) $E \subset W_z$, $N(E)=m$.

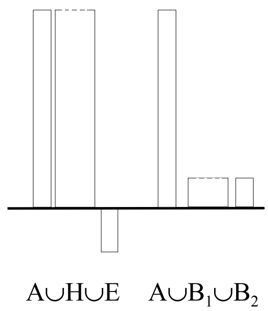

Diagram 5

It follows from the definition of a welfare range that $W_3 \subset R(1, y)$. Accordingly, from (3) we know that $B_1 \subset R(1, y)$. Consequently, from (2), (3), (7), (8), and Condition δ we get that

(9) $A \cup H \cup E$ is at least as good as $A \cup B_1 \cup B_2$ (see *Diagram 5*).

Let

(10) $r > n+p$ be a number of lives which satisfies Condition β for the three welfare levels W_x, W_2, and W_1 and for $n+p$ lives at W_x;
(11) q be any number of lives such that $q \geq m$ and $q \geq r$;
(12) $G \subset W_2$, $N(G)=n+p+r$;
(13) $I \subset W_1$, $N(I)=q-r$;
(14) $F \subset W_1$, $N(F)=r$.

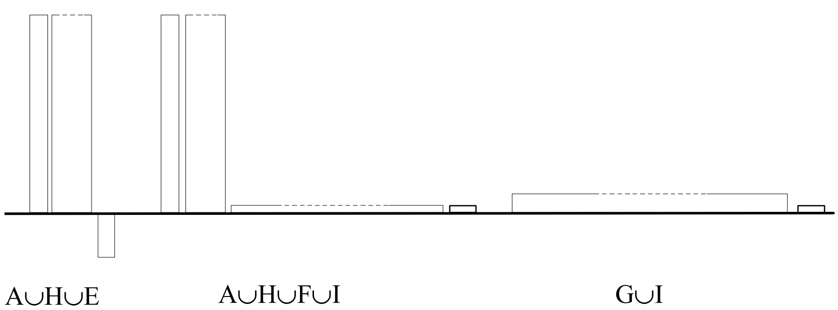

Diagram 6

Since $A \cup H \subset \mathbb{W}_x$, and $N(A \cup H) = n + p$ (by (6) and (7)), and $I \subset \mathbb{R}(1, 3)$, it follows from (10)-(14) and Condition β that

(15) $G \cup I$ is at least as good as $A \cup H \cup F \cup I$ (see *Diagram 6*).

Since the F- and the I-lives have positive welfare (by (13) and (14)), it follows from (1), (8) and the Weak Non-Sadism Condition that

(16) $A \cup H \cup F \cup I$ is at least as good as $A \cup H \cup E$ (see *Diagram 6*).

By transitivity, it follows from (15) and (16) that

(17) $G \cup I$ is at least as good as $A \cup H \cup E$.

Let

(18) $C \subset \mathbb{W}_3$, $N(C) = p + q - m$.

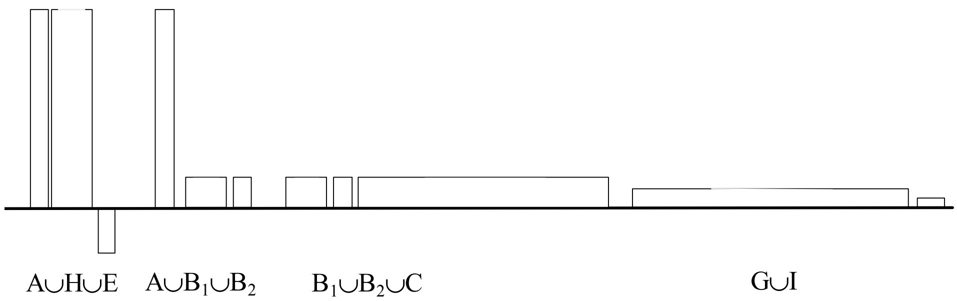

Diagram 7

Since $\mathbb{W}_3 \subset \mathbb{R}(1, v)$, we can conclude that $C \subset \mathbb{R}(1, v)$, and since $x \geq u$ (by (5)), and $A \subset \mathbb{W}_x$ (by (6)), it follows from (4) and the Weak Quality Addition Condition that

(19) $A \cup B_1 \cup B_2$ is at least as good as $B_1 \cup B_2 \cup C$ (see *Diagram 7*).

Since $B_1 \cup B_2 \cup C \subset \mathbb{W}_3$ (by (3) and (18)) and $G \cup I \subset \mathbb{W}_1 \cup \mathbb{W}_2$, (by (12) and (13)) and $N(B_1 \cup B_2 \cup C) = N(G \cup I)$, the Egalitarian Dominance Condition implies that

(20) $G \cup I$ is worse than $B_1 \cup B_2 \cup C$ (see *Diagram 7*).

By transitivity, it follows from (19) and (20) that

(21) $G \cup I$ is worse than $A \cup B_1 \cup B_2$

and from (9) and (21) that

(22) $G \cup I$ is worse than $A \cup H \cup E$

which contradicts (17). Q.E.D.

It follows trivially from lemma 1, 2 and 3 that the impossibility theorem is true.
Q.E.D.

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