0.1 Test of non-linear restrictions - the delta method

Tests do not have to be linear. More generally, a test can be written as

$$H_0 : \mathbf{c}(\boldsymbol{\beta}) = \mathbf{q}$$

where $\mathbf{c}(\cdot)$ is a known function not necessarily linear function.

Use asymptotic theory (the Slutsky theorem) we know that the non-linear function of the estimate is consistent, i.e.

$$\mathrm{p}\lim \mathbf{c}\left(\widehat{\boldsymbol{\beta}}_n\right) = \mathbf{c}\left(\mathrm{p}\lim \widehat{\boldsymbol{\beta}}_n\right) = \mathbf{c}\left(\boldsymbol{\beta}\right)$$

We also need the variance of $\mathbf{c}\left(\mathbf{b}\right)$. To obtain the variance of a non-linear function, we need to use a Taylor expansion $\left(f\left(x\right) = f\left(a\right) + f'\left(a\right)\left(x-a\right)\right)$ around the true parameter vector:

$$\mathbf{c}\left(\mathbf{b}\right) \approx \mathbf{c}\left(\boldsymbol{\beta}\right) + \frac{\partial \mathbf{c}\left(\boldsymbol{\beta}\right)}{\partial \boldsymbol{\beta}}\left(\mathbf{b}-\boldsymbol{\beta}\right)$$

where the first part of the right hand side is a constant. The higher order terms become negligible in large samples since $p\lim \mathbf{b} = \boldsymbol{\beta}$.

$$V\left(\mathbf{c}\left(\mathbf{b}\right)\right) \approx \left(\frac{\partial \mathbf{c}\left(\boldsymbol{\beta}\right)}{\partial \boldsymbol{\beta}}\right)' V\left(\widehat{\boldsymbol{\beta}}\right)\left(\frac{\partial \mathbf{c}\left(\boldsymbol{\beta}\right)}{\partial \boldsymbol{\beta}}\right)$$

$$\left(\mathbf{c}\left(\mathbf{b}\right)-\mathbf{q}\right)'\left(\frac{\partial \mathbf{c}\left(\boldsymbol{\beta}\right)}{\partial \boldsymbol{\beta}}\right)' V\left(\mathbf{b}\right)\left(\frac{\partial \mathbf{c}\left(\boldsymbol{\beta}\right)}{\partial \boldsymbol{\beta}}\right)\left(\mathbf{c}\left(\mathbf{b}\right)-\mathbf{q}\right) \xrightarrow{d} \chi^2(F)$$