ON GLOBAL NON-OscILLATION OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH POLYNOMIAL COEFFICIENTS

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Abstract. In this note we show that a linear ordinary differential equation with polynomial coefficients is globally non-oscillating in $\mathbb{C}P^1$ if and only if it is Fuchsian, and at every its singular point any two distinct characteristic exponents have distinct real parts.

1. Introduction

Let us recall the classical notions of disconjugacy and non-oscillation of a linear ordinary differential equation, see e.g. [3].

Definition 1. A linear ordinary differential equation of order $k$

$$a_k(z)y^{(k)} + a_{k-1}(z)y^{(k-1)} + \ldots + a_0(z)y = 0,$$  \hspace{1cm} (1)

with continuous coefficients $a_j(z)$, $j = 0, \ldots, k$ defined in a neighborhood of some simply-connected subset $I$ of $\mathbb{R}$ or $\mathbb{C}$, is called disconjugate (resp. non-oscillating) in $I$, if every its nontrivial solution has in $I$ at most $k-1$ zeros (resp. finitely many zeros) counted with multiplicities.

Observe that every equation (1) is disconjugate in any sufficiently small interval in $\mathbb{R}$ (resp. any sufficiently small disk in $\mathbb{C}$) centered at an arbitrary point $z_0 \in \mathbb{R}$ (resp. $z_0 \in \mathbb{C}$) such that $a_k(z_0) \neq 0$. Analogously, every equation (1) is non-oscillating in any compact simply-connected set free from the roots of $a_k(z)$.

The study of different aspects and criteria of disconjugacy and non-oscillation has been an active topic in the past. While there exist satisfactory criteria of disconjugacy for the second order equations, the situation with the higher order equations is more complicated. A number of necessary/sufficient conditions of disconjugacy for subsets of $\mathbb{R}$ and $\mathbb{C}$ are known in the literature mostly dating back at least four decades, see e.g. [10], [8], [9]. In the case of equations of order 2, disconjugacy is closely related to Sturm separation theorems; for higher order equations there is a related version of multiplicative Sturmian theory developed in [12].

In this paper, for a linear differential equation with polynomial coefficients, we introduce the notion of its global non-oscillation in $\mathbb{C}P^1$ by which we mean its classical non-oscillation in an arbitrary open contractible domain obtained after the removal from $\mathbb{C}P^1$ of an appropriate cut connecting all the singular points. Although oscillation/non-oscillation in the complex domain have been studied since the 1920’s, (see e.g. [5]), the notion of global non-oscillation seems to be new. As an experienced reader can easily guess, the main motivation for our consideration comes from the second part of Hilbert’s 16th problem.

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Consider a linear homogeneous differential equation
\[ P_k(z)y^{(k)} + P_{k-1}(z)y^{(k-1)} + \ldots + P_0(z)y = 0, \] (2)
with polynomial coefficients \( P_k(z), \ldots, P_0(z) \), and \( \text{GCD}(P_k, P_{k-1}, \ldots, P_0) = 1 \). Let \( S \) be the set of all singular points of (2) in \( \mathbb{C}P^1 \), i.e., the set of all roots of \( P_k(z) \) (together with \( \infty \) if some of the limits \( \lim_{z \to \infty} z^j P_{k-j}(z)/P_k(z) \), \( j = 0, \ldots, k \) is infinite). For a given equation (2), let \( d \) denote the cardinality of \( S \).

**Definition 2.** A system \( \mathcal{C} := \{C_j\}_{j=1}^{d-1} \) of smooth Jordan curves in \( \mathbb{C}P^1 \), each of them connecting a pair of distinct singular points, is called an *admissible cut* for equation (2) if and only if: a) for any \( i \neq j \), the intersection \( C_i \cap C_j \) is either empty or consists of their common endpoint; b) the union \( \bigcup_{j=1}^{d-1} C_j \) is topologically a tree in \( \mathbb{C}P^1 \), i.e., the complement \( \mathbb{C}P^1 \setminus \bigcup_j C_j \) is contractible; c) each \( C_j \) has a well-defined tangent vector at each of its two endpoints.

**Definition 3.** Equation (2) is called *globally non-oscillating* if, for any its admissible cut \( \mathcal{C} \), every its nontrivial solution has finitely many zeros in \( \mathbb{C} \setminus \mathcal{C} \).

The main result of this paper is the following criterion of global non-oscillation.

**Theorem 4.** Equation (2) is globally non-oscillating if and only if:

(i) it is Fuchsian;

(ii) at each singular point all distinct characteristic exponents have pairwise distinct real parts.

**Remark 5.** One can easily notice that (2) is globally non-oscillating if and only if some (and therefore any) domain \( \mathbb{C}P^1 \setminus \mathcal{C} \) can be covered by finitely many open disconjugacy domains. Observe that if one knows such a covering, then one gets an immediate upper bound for the total number of zeros of nontrivial solutions of (2) in \( \mathbb{C}P^1 \setminus \mathcal{C} \). Namely, if the number of covering disconjugacy domains equals \( l \), then any nontrivial solution of (2) has at most \((k-1)l\) zeros in \( \mathbb{C}P^1 \setminus \mathcal{C} \).

In view of Remark 5 the following problem is of fundamental importance.

**Main Problem.** Given an arbitrary equation (2) satisfying the assumptions of Theorem 4, estimate from above the number of disconjugacy domains which can form an open covering of \( \mathbb{C}P^1 \setminus \mathcal{C} \), for some admissible cut \( \mathcal{C} \).

Observe that in case of a Schrödinger equation
\[-y'' + P(z)y = 0\]
with a polynomial potential \( P(z) \), there is a classical construction of such coverings using the Schwarzian derivative of two linearly independent solutions of the latter equation which goes back to R. Nevanlinna, [11].

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2. Proofs

Our proof of Theorem 4 consists of several steps.

Step 1. The necessity of Conditions (i) and (ii) for the global non-oscillation of equation (2).

Indeed, if (2) has a non-Fuchsian singularity at \( p \in \mathbb{C}P^1 \), then, for any sufficiently small \( \epsilon > 0 \), almost any solution of (2) has infinitely many zeros in the \( \epsilon \)-neighborhood of \( p \) with a removed straight segment connecting \( p \) with some point on the bounding circle. This property contradicts to global non-oscillation. To
finish Step 1, consider a Fuchsian singularity of (2) with two distinct characteristic exponents of the form $a + b_1 I$ and $a + b_2 I$. Then there exists a solution of (2) with the leading term $z^{a+\frac{(b_1+b_2)r}{2}} \cos \left( \frac{b_1-b_2}{2} \ln z \right)$. Such a solution has infinitely many zeros accumulating to $p$ which are located close to the horizontal line passing through $p$. This again contradicts to global non-oscillation.

Step 2. Reduction to small neighborhoods of singular points.

For any $\epsilon > 0$ construct the simply-connected domain $U_\epsilon \subset \mathbb{C}$ by cutting by straight lines the complement to the $\epsilon$-neighborhoods of zeros of $P_k$ to a disk $\{ |z| < \epsilon^{-1} \}$.

The following complex analogue of the classical de la Vallée Poussin theorem [4] is proved in [14, Theorem 2.6, Corollary 2.7]. Consider a homogeneous monic linear ordinary differential equation with holomorphic coefficients

$$y^{(k)} + a_1(t) y^{(k-1)} + \cdots + a_k(t) y = 0, \quad t \in \gamma \subset \mathbb{C}.$$ 

Then the variation of argument of any solution $y(t)$ along a circular arc $\gamma$ of known length is explicitly bounded in terms of the uniform upper bounds $A_i = \sup_{t \in \gamma} |a_i(t)|$, $i = 1, \ldots, k$.

This implies an explicit upper bound $B(\epsilon)$ for the number of zeros of any solution of (2) in $U_\epsilon$. More explicit calculations show that $B(\epsilon)$ is polynomial in $\epsilon^{-1}$, with coefficients explicitly depending on (2).

As any admissible system of cuts creates a domain which can be covered by finitely many $U_\epsilon$ and bounded sectors at singular points, the main difficulty consists of providing an upper bounds for the number of zeros of solutions of 2 in these sectors.

Step 3. Equations with constant coefficients. ("Reduction" to this case in a neighborhood of a singular point is obtained by using the logarithmic chart centered at the singularity. See also Step 5.)

**Proposition 6. For any $\alpha \geq 0$ and for any equation**

$$EQ: \quad a_k y^{(k)} + a_{k-1} y^{(k-1)} + \cdots + a_0 y = 0, \quad a_j \in \mathbb{C}, \quad a_k \neq 0 \quad (3)$$

**such that all its distinct characteristic roots have distinct real parts,**

1. **there exists an upper bound $\sharp(EQ, \alpha)$ for the number of zeros in the horizontal strip $\{ \Pi_\alpha : |3(z) \leq \alpha| \}$ for all nontrivial solutions of (3). (Here zeros are counted with multiplicities.)**

2. **If all roots $\lambda_j$ of characteristic equation of (3) are simple, with $\Re \lambda_j < -\Re \lambda_{j+1}$, then**

$$\sharp(EQ, \alpha) \leq (k-1)^2 + \frac{2}{\pi} (k-1) \mathcal{L}(EQ) [\alpha(\Xi + 2) + \Theta \log 4], \quad (4)$$

where $\mathcal{L}(EQ)$ is the length of the shortest polyline passing through all $\lambda_j$ and

$$\Theta = \max_{j=1,\ldots,k-1} |\Re(\lambda_j) - \Re(\lambda_{j+1})|^{-1}, \quad \Xi := \max_{1 \leq j \leq k-1} \left| \frac{3 \lambda_j - 3 \lambda_{j+1}}{\Re \lambda_j - \Re \lambda_{j+1}} \right|.$$

Our approach to the proof of Proposition 6 is inspired by the Wiman-Valiron theory, see [13]. It also has a strong resemblance with constructions in the modern tropical geometry....

The general solution of (3) is given by:

$$y = \sum_j A_j(z)e^{\lambda_j z}, \quad \text{where deg } A_j(z) = n_j, \sum (n_j + 1) = k. \quad (5)$$
Define the \textit{domain of a single term $y$-dominance in $\Pi_\alpha$} as

$$G(y,\alpha) := \{z \in \Pi_\alpha \mid \exists j = j(z), \exists \epsilon > 0 : |A_j(z)e^{\lambda_j z}| \geq (1 - \epsilon) \sum_{i \neq j} |A_i(z)e^{\lambda_i z}|\}. \quad (6)$$

Note that $G(y,\alpha)$ may contain at most $\min n_j \leq k$ zeros of $y$, namely the common zeros of all $A_j(z)$ (in particular, no zeros at all in the case of simple characteristic exponents).

\textbf{Lemma 7.} The complement $\Pi_\alpha \setminus G(y,\alpha)$ can be covered by at most $k + k^2 + k^3$ boxes of total width not exceeding

$$k^2(k + 1)(4\Theta \ln k + 4\alpha \Xi + 4\alpha) + 8k^2\Theta.$$

We deal first with the case of only simple characteristic exponents $\lambda_j$, as this case is much more transparent and the resulting estimates seem to be of right order of magnitude. The polynomials $A_j(z)$ in this case are constants which will be denoted by $a_j$.

\textbf{Lemma 8.} For any $y$ as above, $\Pi_\alpha \setminus G(y,\alpha)$ is contained in the union of at most $k - 1$ horizontal boxes (of height $2\alpha$) of the total width not exceeding

$$2\alpha(k - 1)\Xi + 2(k - 1)\Theta \ln 4. \quad (7)$$

Before proving Lemma 8 and 7 we indicate how they imply Proposition 6. In [6], the following theorem is proved.

\textbf{Theorem 9 ([6])}. Consider a space of quasipolynomials $QP_\Lambda = \{\sum_j A_j(z)e^{\lambda_j z}, A_j \in C[z]\}$ of dimension $k = \sum_k(1 + \deg A_j)$, where $\Lambda = \{\lambda_j\} \subset C$ is some finite set. The number of zeros of any function $f \in QP_\Lambda$ in a bounded convex domain $U$ does not exceed

$$k - 1 + \frac{1}{\pi} \mathcal{L}(\Lambda) \text{diam}(U), \quad (8)$$

where $\mathcal{L}(\Lambda)$ is the length of a shortest polyline passing through all points of $\Lambda$.

This immediately implies an estimate on the number of zeros of $y$ in the boxes $B_j$ of Lemma 8, and, as

$$\sum \text{diam } B_j \leq 2(k - 1)[\alpha \Xi + \Theta \log 4] + 4(k - 1)\alpha,$$

(4) follows.

Almost the same arguments deduce the first part of Proposition 6 from Lemma 7, though we did not bother to write down the exact upper bound.

\textit{Proof of Lemma 8.} The principal case is of $\alpha = 0$, i.e. of $\Pi_0 = \mathbb{R}$.

\textbf{Lemma 10.} In the above notations, $\mathbb{R} \setminus G(y,0)$ is contained in the union of at most $k - 1$ closed intervals of the total length less than or equal to $2(k - 1)\ln 4 \cdot \Theta$.

To prove Lemma 10, we need an additional statement. In $\mathbb{R}^2$ with coordinates $(\mu, \phi)$ consider the 1-parameter family $\{Pt_j(u)\}_{j=1}^k$ of $k$ points given by

$$\mu = \Re(\lambda_j), \quad \phi = \ln|a_je^{\lambda_j u}|,$$

where $u$ is a real-valued parameter. For a given value of $u \in \mathbb{R}$, introduce the piecewise-linear function $\phi_u(\mu)$ as the \textit{least concave majorant} of $\{Pt_j(u)\}_{j=1}^k$. By this we mean the minimal possible concave function $\phi_u(\mu)$ defined in the interval $[\Re(\lambda_1), \Re(\lambda_k)]$ such that all points $\{Pt_j(u)\}_{j=1}^k$ lie non-strictly below its graph, i.e. have their $\phi$-coordinate smaller than or equal to that of $\phi_u(\mu)$. (One can easily see that the graph of $\phi_u(\mu)$ is the upper part of the boundary of the convex hull of $\{Pt_j(u)\}_{j=1}^k$ connecting $Pt_1(u)$ and $Pt_k(u)$.) Observe that, for any $u \in \mathbb{R}$,

$$\phi_u(\mu) = \phi_0(\mu) + u\mu. \quad (9)$$
Lemma 11. If, for \( j = 1, \ldots, k - 1 \),
\[
|\phi_u(\Re(\lambda_{j+1})) - \phi_u(\Re(\lambda_j))| \geq \ln 4,
\]
then \( u \in G(y, 0) \).

Proof of Lemma 11. Define the central index of \( u \) by the formula:
\[
i(u) := \{ i \mid \Re(\lambda_i) \text{ is the point of the global maximum for } \phi_u(\mu) \},
\]
and the statement follows if \( j \neq i \),
\[
|a_j e^{\lambda_j u}| \leq \exp(\phi_u(\Re(\lambda_j))) \leq 4^{-|j-i(u)|}|a_i e^{\lambda_i u}|.
\]
Therefore the inequality in the definition (6) of \( G(y, 0) \) follows after the summation of a geometric series.

Corollary 12. If \(-u \) lies outside the \( \ln 4 \cdot \Theta \)-neighborhood of the set of all slopes of \( \phi_0(\mu) \), then \( u \in G(y, 0) \).

Proof. Formula (9) implies that each slope of \( \phi_u(\mu) \) equals the sum of the respective slope of \( \phi_0(\mu) \) and \( u \). Therefore in the considered case, all slopes of \( \phi_u(\mu) \) exceed in absolute value the number \( \ln 4 \cdot \Theta \), and the statement follows immediately from Lemma 11.

Proof of Lemma 10. The \( \ln 4 \cdot \Theta \)-neighborhood of the set of slopes of \( \phi_0(\mu) \) consists of the union of at most \( k-1 \) intervals of total length not exceeding \( 2(k-1) \ln 4 \cdot \Theta \).

Now, consider the general case of \( \Pi_\alpha \), \( \alpha \geq 0 \).

We repeat the above construction of Lemma 10 for \( z \) running along the horizontal line \( 3z = v \) with \( |v| \leq \alpha \). For every fixed \( v \), consider in \( \mathbb{R}^2 \) with coordinates \( (\mu, \phi) \), the 1-parameter family \( \{ P_\mu^v(u) \}_j^k \) of \( k \) points given by
\[
\mu = \{ \Re(\lambda_j), \phi = \ln |a_j e^{\lambda_j (u+iv)}| \}
\]
where \( u \) is a real parameter. Introduce \( \phi_v^u(\mu) \) as the least concave majorant of \( \{ P_\mu^v(u) \}_j^k \), for a given value of \( u \in \mathbb{R} \). Observe that, for any \( u \in \mathbb{R} \),
\[
\phi_v^u(\mu) = \phi_0^u(\mu) + u \mu. \tag{10}
\]
Now consider the set \( S_{\Pi_\alpha} := \cup_{\alpha \leq \varepsilon \leq \alpha} \{ k_j(v) \} \), where \( k_j(v) \) are the slopes of \( \phi_v^u(\mu) \). We claim that \( S_{\Pi_\alpha} \) is the union of at most \( k-1 \) closed intervals. Indeed, the set of slopes \( \{ k_j(v) \} \) changes continuously with \( v \), and consists of no more than \( k-1 \) points for each fixed \( v \).

Moreover, as \( \ln |a_j e^{\lambda_j v}| = \ln |a_j| - v \Im(\lambda_j) \), the points \( \{ P_\mu^v(0) \}_j^k \) defining \( \phi_v^u(\mu) \) depend linearly on \( v \), namely they move up or down as \( v \) changes. The inequality
\[
\left| \frac{\partial k_j(v)}{\partial v} \right| \leq \Xi
\]
is straightforward. Therefore the total length of \( S_{\Pi_\alpha} \) is at most \( 2\alpha \Xi(k-1) \).

By Corollary 12, if \(-u \) lies outside the \( \ln 4 \cdot \Theta \)-neighborhood of \( S_{\Pi_\alpha} \), then, for any \( |v| \leq \alpha \), \( u + iv \) lies in \( G(y, \alpha) \) which settles Lemma 8.

Step 6. Case of multiple characteristic exponents.

In this case the dependence on \( v \) of (analogues of) points \( P_\mu^v(u) \) seems to be more complicated, and we are forced to consider the slopes of all chords joining them, not only lying on the boundary of their convex hull. This leads to a conjecturally excessive bound.
Proof. Consider the absolute value $r_{jj'}$ of the ratio of any two terms in (5). The complement $\Pi_\alpha \setminus G(y, \alpha)$ lies in the union $\Sigma$ of the sets $\Sigma_{jj'}^\alpha = \{|\ln r_{jj'}(z)| \leq \ln k\}$, where $r_{jj'}$ is the absolute value of the ratio of two terms in (5).

We can write

$$\ln r_{jj'} = \ln |A_j/A_{j'}| - v\xi_{jj'}\theta_{jj'} + \theta_{jj'}u,$$

where

$$\theta_{jj'} = \Re(\lambda_j - \lambda_{j'}), \quad \xi_{jj'} = \theta_{jj'}^{-1}\Im(\lambda_j - \lambda_{j'}),$$

Let $W = \{ |\Re(z - z_i) | \geq 4k\Theta \}$, where $z_i$ runs over all roots of all $A_j$. Outside $W$ we have

$$\left| \frac{\partial}{\partial u} \ln |A_j/A_{j'}| \right|, \left| \frac{\partial}{\partial v} \ln |A_j/A_{j'}| \right| \leq \frac{|\theta_{jj'}|}{2}.$$

It follows that

$$\Sigma_{jj'}^\alpha \subset \Sigma_{jj'} = \{ u + iv \in \Pi_\alpha : |\ln r_{jj'}(u)| \leq \ln k + \alpha|\xi_{jj'}\theta_{jj'} + \alpha|\theta_{jj'}| \}.$$

Note that $\Sigma_{jj'}$ is a union of boxes, as its definition does not depend on $v$.

It follows that outside $W$

$$\left| \frac{\partial}{\partial u} \ln r_{jj'} \right| \geq \frac{|\theta_{jj'}|}{2}. \quad (12)$$

Therefore $\Sigma_{jj'}$ intersects each connected component of $\mathbb{R} \setminus W$ in an interval of length at most $4|\theta_{jj'}|^{-1}\ln k + 4\alpha|\xi_{jj'}| + 4\alpha$. In other words, $\Sigma_{jj'} \setminus W$ is the union of at most $k+1$ boxes of total width not exceeding $(k+1)(4\ln k)|\theta_{jj'}|^{-1} + 4\alpha|\xi_{jj'}| + 4\alpha$.

Taking the union over all possible pairs $(j, j')$, we conclude that $\Sigma \setminus W$ lies in the union of at most $k^2(k+1)$ boxes of total width at most $k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha)$. As $W \cap \Pi_\alpha$ is the union of at most $k$ boxes of width at most $8k\Theta$ each, we obtain that $\Sigma$ lies in the union of at most $k + k^2 + k^3$ boxes of total width at most $k^2(k+1)(4\Theta \ln k + 4\alpha\Xi + 4\alpha) + 8k^2\Theta$. \hfill $\Box$

Step 5. Equation with non-constant coefficients in a semistrip.

In general, solutions of (2) considered in the logarithmic chart near its Fuchsian point have the form

$$y = \sum_j \hat{A}_j(z)e^{\lambda_jz}, \quad (13)$$

where

$$\hat{A}_j(z) = \sum_{r=0}^{n_j} a_{j,r}z^{n_j-r}(1 + \epsilon_{j,r}),$$

and $\epsilon_{j,r} = O(e^r)$ in any semistrip $\Pi_{\alpha,\beta} = \{|\Im z| \leq \alpha, \Re z \leq \beta \}$ and depend on (2) only.

Lemma 13. Let $A_j(z) = \sum_{r=0}^{n_j} a_{j,r}z^{n_j-r}$ and $W$ be as in the proof of Lemma 7. Then

$$|\log |\hat{A}_j/A_j| | \leq ???(e) \text{ in } W \quad (14)$$

if $|\epsilon_r| < e$.

PROOF????

Lemma 14. For a given $\alpha$, choose a sufficiently small $\beta$ such that

$$|\ln(1 + \epsilon_r(z))| \leq 0.1 \min (\Re \lambda_{j+1} - \Re \lambda_j) \text{ in } \Pi_{\alpha,\beta}. \quad (15)$$

Then zeros of (13) in $\Pi_{\alpha,\beta}$ lie in the union of at most $k - 1$ horizontal boxes (with the height $2\alpha$) of the total width not exceeding

$$2\alpha(k - 1)\Xi + 2(k - 1)\Theta \ln 4.$$
Proof. Indeed, if the slopes of the least concave majorant \( \phi_v(\mu) \) were at least ln 4 in absolute value then the slopes of the least concave majorant of the set of points \( \{\Re(\lambda_j), \ln |a_j(1 + \epsilon_j(z))e^{\lambda_j(u + Iv)}|\} \) will have the absolute values of slopes greater than ln 3, which is sufficient to guarantee that one of summands in (13) majorizes all others. So the same estimate as in Lemma 8 holds. \( \square \)

Let \( y^{(k)} + b_1(z)y^{(k-1)} + \ldots + b_k y = 0 \) be the reduced form (=divided by its leading term) of (2) in the logarithmic chart near its singular Fuchsian point. Assume that \( b_j(z) \) are bounded by \( C \) in \( \Pi_{\alpha, \beta} \) (Fuchsianity implies that \( b_j(z) \) tends to some finite limit as \( z \to \infty \) in \( \Pi_{\alpha, \beta} \)).

Example in [14] immediately following Corollary 2.7 then implies that \( y \) has at most \( 2(k+1) + \frac{k+1}{6\Theta\ln 4}C \) in \( \Pi_{\alpha, \beta} \), where \( \ell \leq 2(2\alpha(k-1)\Xi + 2(k-1)\Theta \ln 4) + 4(k-1)\alpha \) is the total perimeter of all boxes of Lemma 14.

After going back from logarithmic chart, this provides an upper bound for the number of zeros of any solution of (2) in the sector \( \{|z-p| \leq e^{\beta}, |\arg z| \leq \alpha\} \) at the Fuchsian singular point \( p \).

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